This exam contains twenty multiple-choice problems worth five points each for an exam total of 100 points. For each problem, mark your answer card with the letter corresponding to the only correct answer.

You may refer to the following table during the exam.

**Table of Laplace Transforms**

| $f(t)$ | $F(s)$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $\sin at$ | $\frac{a}{s^2+a^2}$ |
| $\cos at$ | $\frac{s}{s^2+a^2}$ |
| $\sinh at$ | $\frac{a}{s^2-a^2}$ |
| $\cosh at$ | $\frac{s}{s^2-a^2}$ |
| $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}$ |
| $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ |
| $u_c(t)$ | $\frac{e^{-cs}}{s}$ |
| $u_c(t) \cdot f(t-c)$ | $e^{-cs}F(s)$ |
| $\delta(t-c)$ | $e^{-cs}$ |

1. Find the general solution of the following differential equation, and determine how the solutions behave as $t \to \infty$.

$$ty' + 2y = 6, \qquad t > 0$$

(A) $y \to -\infty$

(B) $y \to -6$

(C) $y \to -3$

(D) $y \to -2$

(E) $y \to 0$

(F) $y \to 2$

(G) $y \to 3$

(H) $y \to 6$

(I) $y \to \infty$

(J) The limit cannot be determined because it depends on the initial conditions.

$$y' + \frac{2}{t}y = \frac{6}{t} \qquad \mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2 y' + 2ty = 6t$$

$$\int [t^2 y' + 2ty]\,dt = \int 6t\,dt$$

$$t^2 y = 3t^2 + C$$

$$y = 3 + \frac{C}{t^2}$$

as $t \to \infty$, $y \to 3$

2. Consider the initial value problem $(t^2 - 3t)y'' + ty' - (t+3)y = 0$, $y(1) = 2$, $y'(1) = 1$. The existence and uniqueness theorem for linear differential equations guarantees that there will be a unique solution. Determine the longest interval in which this solution is certain to exist.

(A) $-\infty < t < 0$

(B) $-\infty < t < 2$

(C) $-\infty < t < 3$

(D) $-\infty < t < \infty$

(E) $0 < t < 2$

(F) $0 < t < 3$

(G) $0 < t < \infty$

(H) $2 < t < 3$

(I) $2 < t < \infty$

(J) $3 < t < \infty$

$$y'' + \frac{t}{t^2-3t}y' - \frac{t+3}{t^2-3t}y = 0 \qquad t_0 = 1$$

discontinuous at $t = 0$ and $t = 3$

the longest interval containing $t_0 = 1$ throughout which the coefficient functions are continuous: $(0, 3)$

3. Find the general solution of the following differential equation.

$$y'' + 6y' + 9y = 0$$

$$r^2 + 6r + 9 = 0$$

$$r = -3, -3$$

(A) $y = c_1 e^{3t} + c_2 e^{3t}$

(B) $y = c_1 e^{3t} + c_2 t^3 e^{3t}$

(C) $y = c_1 e^{-3t} + c_2 e^{-3t}$

(D) $y = c_1 e^{-3t} + c_2 t e^{-3t}$

(E) $y = c_1 e^{3t} + c_2 e^{-3t}$

(F) $y = c_1 t^3 + c_2 t^3$

(G) $y = c_1 t^3 + c_2 t^3 \ln t$

(H) $y = c_1 t^{-3} + c_2 t^{-3}$

(I) $y = c_1 t^{-3} + c_2 t^{-3} \ln t$

(J) $y = c_1 t^3 + c_2 t^{-3}$

4. Consider a spring system in which a 16 pound object stretches the spring six inches. (There is no damping.) What external force would cause resonance to occur?

$16 = m \cdot 32$ $\qquad$ $16 = k \cdot \frac{1}{2}$

$m = \frac{1}{2}$ $\qquad$ $k = 32$

$$\frac{1}{2} u'' + 32u = 0$$

$$u'' + 64u = 0$$

$$r^2 + 64 = 0$$

$$r = \pm 8i$$

$$u = c_1 \cos 8t + c_2 \sin 8t$$

natural frequency: 8

(A) $F(t) = t$

(B) $F(t) = e^t$

(C) $F(t) = te^t$

(D) $F(t) = e^{8t}$

(E) $F(t) = e^{-8t}$

(F) $F(t) = 2\cos(\sqrt{2}t)$

(G) $F(t) = \sin(\frac{4}{\sqrt{3}}t)$

(H) $F(t) = 5\cos(\frac{1}{4}t)$

(I) $F(t) = 6\sin(\frac{1}{\sqrt{6}}t)$

(J) $F(t) = 3\cos(8t) - \sin(8t)$

5. Consider the differential equation $2x(x-5)^2y'' + xy' + (x-5)y = 0$. Classify the point $x_0 = 3$.

(A) It is an ordinary point of the differential equation.

(B) It is a regular singular point of the differential equation.

(C) It is an irregular singular point of the differential equation.

$P(x) = 2x(x-5)^2$

$P(3) \neq 0$

6. Consider the differential equation $2x(x-5)^2y'' + xy' + (x-5)y = 0$. Determine a lower bound for the radius of convergence of the series solutions about $x_0 = 3$ for this differential equation.

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(F) 5

(G) 6

(H) 7

(I) 8

(J) $\infty$

$P(x) = 2x(x-5)^2$

zeros of $P$: $x = 0$ and $x = 5$

distance from $x_0 = 3$ to the nearest zero of $P$: 2

7. Consider the differential equation $xy'' - 4xy' + 4y = 0$. Note that $x_0 = 0$ is a regular singular point. Find the form of two linearly independent series solutions about $x_0 = 0$. (Hint: In the book, there are some forms in which "$b_n$" is used for the coefficients in the second solution and some forms in which "$c_n$" is used. However, it is not important which variable is used. For the sake of consistency, "$b_n$" is used in each of the answers below.)

(A) $y_1 = \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = \sum_{n=0}^{\infty} b_n x^n$

(B) $y_1 = \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n$

(C) $y_1 = \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = y_1 \ln x + x \sum_{n=0}^{\infty} b_n x^n$

(D) $y_1 = x \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = \sum_{n=0}^{\infty} b_n x^n$

(E) $y_1 = x \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = a y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n$

(F) $y_1 = x \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = y_1 \ln x + x^4 \sum_{n=0}^{\infty} b_n x^n$

(G) $y_1 = x^4 \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = x \sum_{n=0}^{\infty} b_n x^n$

(H) $y_1 = x^4 \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = a y_1 \ln x + x \sum_{n=0}^{\infty} b_n x^n$

(I) $y_1 = x^2 \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = x^2 \sum_{n=0}^{\infty} b_n x^n$

(J) $y_1 = x^2 \sum_{n=0}^{\infty} a_n x^n \qquad y_2 = y_1 \ln x + x^2 \sum_{n=1}^{\infty} b_n x^n$

$\lim_{x \to 0} x \cdot \frac{-4x}{x} = 0$

$\lim_{x \to 0} x^2 \cdot \frac{4}{x} = 0$

$r(r-1) + 0r + 0 = 0$

$r = 0, 1$

$r_1 = 1, \quad r_2 = 0$

roots differ by an integer

8. Consider the differential equation $x^2y'' + xy' + xy = 0$. It turns out that there is a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. (You do not need to verify this.) Set ~~$a_0 = 0$~~, and find $a_1$.

$a_0 = 1$

(A) $-1$

(B) $-\frac{1}{2}$

(C) $-\frac{1}{3}$

(D) $-\frac{1}{4}$

(E) $-\frac{1}{6}$

(F) $-\frac{1}{9}$

(G) $-\frac{1}{12}$

(H) $-\frac{1}{18}$

(I) $-\frac{1}{36}$

(J) 0

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$xy' = \sum_{n=0}^{\infty} n a_n x^n = \sum_{n=1}^{\infty} n a_n x^n$$

$$x^2 y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^n = \sum_{n=1}^{\infty} n(n-1) a_n x^n$$

$$\sum_{n=1}^{\infty} \left[ n(n-1)a_n + n a_n + a_{n-1} \right] x^n = 0$$

For $n \geq 1$, $n(n-1)a_n + n a_n + a_{n-1} = 0$

$$n^2 a_n = -a_{n-1}$$

$$a_n = \frac{-a_{n-1}}{n^2}$$

If $a_0 = 1$, $a_1 = -1$.

9. Let $f(t) = \begin{cases} 0, & 0 \le x < 3 \\ 1, & 3 \le x < 6 \\ 0, & x \ge 6 \end{cases}$. Find $\mathcal{L}\{f(t)\}$.

$$f(t) = u_3(t) - u_6(t)$$

$$\mathcal{L}\{f(t)\} = \frac{e^{-3s}}{s} - \frac{e^{-6s}}{s}$$

(A) $e^{-3s} + e^{-6s}$

(B) $e^{-3s} - e^{-6s}$

(C) $e^{-6s} - e^{-3s}$

(D) $\frac{e^{-3s}}{s}$

(E) $\frac{e^{-6s}}{s}$

(F) $\frac{6e^{-3s}}{s}$

(G) $\frac{3e^{-6s}}{s}$

(H) $\frac{e^{-3s}}{s} + \frac{e^{-6s}}{s}$

(I) $\frac{e^{-3s}}{s} - \frac{e^{-6s}}{s}$

(J) $\frac{e^{-6s}}{s} - \frac{e^{-3s}}{s}$

10. Find $\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{8}{s^2+4}\right)\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{8}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{4 \cdot \frac{2}{s^2+4}\right\}$$

$$= 4\sin 2t$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{8}{s^2+4}\right)\right\}$$

$$= u_3(t) \cdot 4\sin 2(t-3)$$

(A) $u_3(t) \cdot 4\sin 2t$

(B) $u_3(t) \cdot 8\sin 2t$

(C) $u_3(t) \cdot 4\sin 2(t-3)$

(D) $u_3(t) \cdot 8\sin 2(t-3)$

(E) $\delta(t-3) \cdot 4\sin 2t$

(F) $\delta(t-3) \cdot 8\sin 2t$

(G) $\delta(t-3) \cdot 4\sin 2(t-3)$

(H) $\delta(t-3) \cdot 8\sin 2(t-3)$

11. For the matrix $\begin{pmatrix} 2 & 1 & 4 \\ 9 & 2 & 5 \\ 0 & 0 & 1 \end{pmatrix}$, $\lambda = 5$ is an eigenvalue with associated eigenvector $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$.

(You do not need to verify this.) What eigenvalue is associated with the vector $\begin{pmatrix} 7.5 \\ 22.5 \\ 0 \end{pmatrix}$?

(A) $-1$
(B) 0
(C) 1
(D) 5
(E) 7.5
(F) 15
(G) 22.5
(H) 25
(I) 37.5
(J) None, because the vector is not an eigenvector.

The second vector is a constant multiple of the first, so it is an eigenvector for the same eigenvalue $\lambda = 5$. (This can also be checked by direct multiplication of the matrix and the vector.)

12. Find the general solution of the following system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$$

(A) $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$

(B) $\mathbf{x} = c_1\begin{pmatrix} -1 \\ 3 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{4t}$

(C) $\mathbf{x} = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{4t}$

(D) $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{5t}$

(E) $\mathbf{x} = c_1\begin{pmatrix} 0 \\ 3 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 4 \\ 3 \end{pmatrix}e^{5t}$

(F) $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 4 \\ 3 \end{pmatrix}e^{5t}$

(G) $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 7 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ 9 \end{pmatrix}e^{-4t}$

(H) $\mathbf{x} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 5 \\ -3 \end{pmatrix}e^{-4t}$

(I) $\mathbf{x} = c_1\begin{pmatrix} 1 \\ 6 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 10 \end{pmatrix}e^{-5t}$

(J) $\mathbf{x} = c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 4 \\ -3 \end{pmatrix}e^{-5t}$

$$\det\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda = 2, 4$$

$$\lambda = 2: \quad \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\lambda = 4: \quad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

13. Each of the following is the general solution of a system of differential equations. Consider the phase portrait for each. For which one(s) is the origin a node?

(I) $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ one negative, one positive eigenvalue saddle point

(II) $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$ two negative eigenvalues

(III) $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ two positive eigenvalues

$\left.\begin{matrix} \text{two negative eigenvalues} \\ \text{two positive eigenvalues} \end{matrix}\right\} \text{node}$

(A) I, II, and III

(B) I and II only

(C) I and III only

(D) II and III only

(E) I only

(F) II only

(G) III only

(H) none of these

14. Consider the following matrix.

$$A = \begin{pmatrix} 0 & 211 & 315.5 & 404 & 6\pi \\ 211 & 0 & -17 & -0.5 & 3e \\ 315.5 & -17 & K & 100 & -1 \\ 404 & -0.5 & 100 & 0 & 2 \\ 6\pi & 3e & -1 & 2 & 0 \end{pmatrix}$$

(Assume that $K$ is real.)

Suppose $S$ is a maximal set of linearly independent eigenvectors for $A$. (In other words, $S$ contains as many eigenvectors as possible.) How many vectors must $S$ contain? (Hint: This problem requires understanding, not computation.)

(A) exactly 1

(B) exactly 2

(C) exactly 3

(D) exactly 4

(E) exactly 5

The matrix is a real symmetric matrix, so it will have a sufficient number (5) of eigenvectors even if there is a deficiency of eigenvalues.

(F) The number of elements in $S$ could be anything from 1 to 5, depending on the value of $K$.

15. Consider the following system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \mathbf{x}$$

The eigenvalues and corresponding eigenvectors for the matrix are as follows.

$$\lambda_1 = 1 + 3i \qquad \xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda_2 = 1 - 3i \qquad \xi^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(You do not need to verify this.) Find the general solution of this system, expressed in terms of real-valued functions.

(A) $\mathbf{x} = c_1 e^{3t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$

(B) $\mathbf{x} = c_1 e^{3t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

(C) $\mathbf{x} = c_1 e^{3t} \begin{pmatrix} \cos t \\ \cos t \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \sin t \\ -\sin t \end{pmatrix}$

(D) $\mathbf{x} = c_1 e^{3t} \begin{pmatrix} \cos t \\ -\cos t \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \sin t \\ \sin t \end{pmatrix}$

(E) $\mathbf{x} = c_1 e^{t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \sin 3t \\ -\cos 3t \end{pmatrix}$

(F) $\mathbf{x} = c_1 e^{t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$

(G) $\mathbf{x} = c_1 e^{t} \begin{pmatrix} \cos 3t \\ \cos 3t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \sin 3t \\ -\sin 3t \end{pmatrix}$

(H) $\mathbf{x} = c_1 e^{t} \begin{pmatrix} \cos 3t \\ -\cos 3t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \sin 3t \\ \sin 3t \end{pmatrix}$

$$X^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+3i)t}$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{t}(\cos 3t + i \sin 3t)$$

$$= e^{t} \begin{pmatrix} \cos 3t + i \sin 3t \\ i \cos 3t - \sin 3t \end{pmatrix}$$

$$= e^{t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + i \cdot e^{t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

$$u(t) = e^{t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$$

$$v(t) = e^{t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

16. Consider the following nonhomogeneous system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ -2t^{-3} \end{pmatrix}$$

The general solution of the corresponding homogeneous system is as follows.

$$\mathbf{x}_c = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} t \\ 2t-1 \end{pmatrix}$$

(You do not need to verify this.) Find $\mathbf{v}(t)$, a particular solution of the nonhomogeneous system. (Hint: Due to the form of $\mathbf{g}(t) = \begin{pmatrix} 0 \\ -2t^{-3} \end{pmatrix}$, the method of undetermined coefficients cannot be used. In addition, the coefficient matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ is not diagonalizable, so the method of diagonalization cannot be used. Therefore the method of variation of parameters must be used.)

(A) $\mathbf{v}(t) = \begin{pmatrix} 0 \\ 6t^{-4} \end{pmatrix}$

(B) $\mathbf{v}(t) = \begin{pmatrix} 0 \\ t^{-2} \end{pmatrix}$

(C) $\mathbf{v}(t) = \begin{pmatrix} 0 \\ t^{-2} + 2t^{-3} \end{pmatrix}$

(D) $\mathbf{v}(t) = \begin{pmatrix} t^{-1} + \ln t \\ 0 \end{pmatrix}$

(E) $\mathbf{v}(t) = \begin{pmatrix} t^{-1} + \ln t \\ \ln t \end{pmatrix}$

(F) $\mathbf{v}(t) = \begin{pmatrix} 2t^{-1} \\ -t^{-2} \end{pmatrix}$

(G) $\mathbf{v}(t) = \begin{pmatrix} t^{-2} \\ -2t^{-1} \end{pmatrix}$

(H) $\mathbf{v}(t) = \begin{pmatrix} -2t^{-2} \\ 2t^{-3} \end{pmatrix}$

(I) $\mathbf{v}(t) = \begin{pmatrix} t^{-1} \\ 2t^{-1} + t^{-2} \end{pmatrix}$

(J) $\mathbf{v}(t) = \begin{pmatrix} -t^{-1} \\ 2t^{-2} - t^{-3} \end{pmatrix}$

$$\Psi(t) = \begin{pmatrix} 1 & t \\ 2 & 2t-1 \end{pmatrix}$$

$$\Psi^{-1}(t) = \frac{1}{-1}\begin{pmatrix} 2t-1 & -t \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2t+1 & t \\ 2 & -1 \end{pmatrix}$$

$$\Psi^{-1}(t)\cdot g(t) = \begin{pmatrix} -2t+1 & t \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -2t^{-3} \end{pmatrix}$$

$$= \begin{pmatrix} -2t^{-2} \\ 2t^{-3} \end{pmatrix} = u'(t)$$

$$u(t) = \begin{pmatrix} 2t^{-1} \\ -t^{-2} \end{pmatrix}$$

$$v(t) = \Psi(t)\cdot u(t) = \begin{pmatrix} 1 & t \\ 2 & 2t-1 \end{pmatrix}\begin{pmatrix} 2t^{-1} \\ -t^{-2} \end{pmatrix}$$

$$= \begin{pmatrix} 2t^{-1} - t^{-1} \\ 4t^{-1} - 2t^{-1} + t^{-2} \end{pmatrix}$$

$$= \begin{pmatrix} t^{-1} \\ 2t^{-1} + t^{-2} \end{pmatrix}$$

17. Find all solutions of the following boundary value problem.

$$y'' + 4y = 0, \quad y(0) = 1, \quad y(\pi) = 1$$

(A) $y = 0$

(B) $y = 1$

(C) $y = \cos 2x$

(D) $y = \sin 2x$

(E) $y = \cos 2x + \sin 2x$

(F) $y = c_1 \cos 2x$, where $c_1$ is arbitrary

(G) $y = c_2 \sin 2x$, where $c_2$ is arbitrary

(H) $y = c_1 \cos 2x + \sin 2x$, where $c_1$ is arbitrary

(I) $y = \cos 2x + c_2 \sin 2x$, where $c_2$ is arbitrary

(J) no solution

$r^2 + 4 = 0 \quad r = \pm 2i$

$y = c_1 \cos 2x + c_2 \sin 2x$

$y(0) = 1 : \quad 1 = c_1$

$y = \cos 2x + c_2 \sin 2x$

$y(\pi) = 1 : \quad 1 = 1$

$c_2$ remains arbitrary

$y = \cos 2x + c_2 \sin 2x$

18. Exactly two of the following statements are false. Which two are they?

T (I) If $g(x)$ is an even function, then it is symmetric with respect to the $y$-axis.

F (II) $f(x) = x^3 + \cos x$ is an odd function.

T (III) $f(x) = x^3 \cdot \cos x$ is an odd function.

F (IV) If $g(x)$ is an even function, then $\int_{-L}^{L} g(x)dx = 0$.

T (V) If $g(x)$ is an odd function having a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$, then $a_n = 0$ for all $n$.

(A) I and II

(B) I and III

(C) I and IV

(D) I and V

(E) II and III

(F) II and IV

(G) II and V

(H) III and IV

(I) III and V

(J) IV and V

19. Let $f(x) = \begin{cases} 0, & -2 \le x < 0 \\ 1, & 0 \le x < 2 \end{cases}$, $\quad f(x+4) = f(x) \qquad L = 2$

Suppose the Fourier series for $f$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$. Compute the coefficients $b_n$.

(A) $b_n = 0$ for all $n$

(B) $b_n = \begin{cases} 0, & n \text{ even} \\ \frac{1}{n\pi}, & n \text{ odd} \end{cases}$

(C) $b_n = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}, & n \text{ odd} \end{cases}$

(D) $b_n = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}$

(E) $b_n = \begin{cases} 0, & n \text{ even} \\ \frac{8}{n\pi}, & n \text{ odd} \end{cases}$

(F) $b_n = \begin{cases} \frac{1}{n\pi}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

(G) $b_n = \begin{cases} \frac{2}{n\pi}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

(H) $b_n = \begin{cases} \frac{4}{n\pi}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

(I) $b_n = \begin{cases} \frac{8}{n\pi}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{0}^{2} \sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{-2}{n\pi}\right)\cos\frac{n\pi x}{2}\Big|_0^2$$

$$= \frac{-1}{n\pi}\left[\cos(n\pi) - 1\right]$$

$$= \begin{cases} \frac{-1}{n\pi}(0), & n \text{ even} \\ \frac{-1}{n\pi}(-2), & n \text{ odd} \end{cases}$$

20. Solve the following heat conduction partial differential equation.

$$9u_{xx} = u_t, \quad u(0,t) = 0, \quad u(3,t) = 0, \quad u(x,0) = \sin 2\pi x$$

(A) $u(x,t) = e^{-4\pi^2 t} \sin 6\pi x$

(B) $u(x,t) = e^{-4\pi^2 t} \sin \frac{\pi x}{9}$

(C) $u(x,t) = e^{-36\pi^2 t} \sin 2\pi x$

(D) $u(x,t) = e^{-36\pi^2 t} \sin 6\pi x$

(E) $u(x,t) = e^{-108\pi^2 t} \sin \frac{\pi x}{3}$

(F) $u(x,t) = e^{-108\pi^2 t} \sin 2\pi x$

(G) $u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin \frac{n\pi x}{3}$

(H) $u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin \frac{n\pi x}{9}$

(I) $u(x,t) = \sum_{n=1}^{\infty} e^{-3n^2\pi^2 t} \sin 2\pi x$

(J) $u(x,t) = \sum_{n=1}^{\infty} e^{-3n^2\pi^2 t} \sin 6\pi x$

$\alpha^2 = 9 \qquad L = 3$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin \frac{n\pi x}{3}$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{3} = \sin 2\pi x$$

$n = 6 : \quad C_6 = 1$

$C_n = 0$ for $n \neq 6$

$$u(x,t) = e^{-36\pi^2 t} \sin 2\pi x$$