This exam contains ten multiple-choice problems worth two points each, five true-false problems worth one point each, and five computational problems worth 25 points altogether, for an exam total of 50 points.

You may not use a calculator of any kind on this exam.

You may refer to the following table during the exam.

### Table of Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s}, s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}, s &gt; a$</td>
</tr>
<tr>
<td>$t^n$ (for a positive integer $n$)</td>
<td>$\frac{n!}{s^{n+1}}, s &gt; 0$</td>
</tr>
<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2+a^2}, s &gt; 0$</td>
</tr>
<tr>
<td>$\cos at$</td>
<td>$\frac{s}{s^2+a^2}, s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}\sin bt$</td>
<td>$\frac{b}{(s-a)^2+b^2}, s &gt; a$</td>
</tr>
<tr>
<td>$e^{at}\cos bt$</td>
<td>$\frac{s-a}{(s-a)^2+b^2}, s &gt; a$</td>
</tr>
<tr>
<td>$t^n e^{at}$ (for a positive integer $n$)</td>
<td>$\frac{n!}{(s-a)^{n+1}}, s &gt; a$</td>
</tr>
<tr>
<td>$u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}, s &gt; 0$</td>
</tr>
<tr>
<td>$u_c(t) \cdot f(t-c)$</td>
<td>$e^{-cs}F(s)$</td>
</tr>
<tr>
<td>$\delta(t-c)$</td>
<td>$e^{-cs}$</td>
</tr>
</tbody>
</table>
Part I. Multiple Choice. (2 points each)

For each of the following, mark your answer card with the letter corresponding to the only correct answer.

1. A spring is stretched six inches by a two pound object. The object is pushed up three inches and then released. There is no damping or external force. Find the period and amplitude of the motion.

\( m = \frac{2}{32} = \frac{1}{16} \)
\( 2 = k \left( \frac{1}{2} \right) \)
\( k = 4 \)

\( \frac{1}{16} u'' + 4u = 0 \quad u'' + 64u = 0 \)

\( r^2 + 64 = 0 \quad r = \pm 8i \)

\[ u = c_1 \cos 8t + c_2 \sin 8t \quad u(0) = -\frac{1}{4} \quad u'(0) = 0 \]

\( -\frac{1}{4} = c_1 \quad u = -\frac{1}{4} \cos 8t + c_2 \sin 8t \)

\[ u' = 2 \sin 8t + 8c_2 \cos 8t \quad 0 = 8c_2 \quad c_2 = 0 \]

\[ u = -\frac{1}{4} \cos 8t \]

2. Suppose that \( y_1 \) is a solution of a linear nonhomogeneous differential equation with constant coefficients \( ay'' + by' + cy = g(t) \), and that \( y_2 \) is a solution of the corresponding homogeneous differential equation \( ay'' + by' + cy = 0 \). Then exactly two of the following are solutions to the nonhomogeneous equation. Which two are they?

(I) \( y_1 + y_2 \)
(II) \( 2y_1 \)
(III) \( 2y_2 \)
(IV) \( 2y_1 + y_2 \)
(V) \( y_1 + 2y_2 \)

(A) (I) and (II)
(B) (I) and (III)
(C) (I) and (IV)
(D) (I) and (V)
(E) (II) and (III)
(F) (II) and (IV)
(G) (II) and (V)
(H) (III) and (IV)
(I) (III) and (V)
(J) (IV) and (V)
3. Find the general solution of the following linear homogeneous differential equation.

\[ x^2y'' - 4xy' + 4y = 0 \]

(A) \[ y = c_1 e^{-2x} + c_2 xe^{-2x} \]

(B) \[ y = c_1 e^{-x} + c_2 e^{-4x} \]

(C) \[ y = c_1 e^x + c_2 e^{4x} \]

(D) \[ y = c_1 e^{2x} + c_2 xe^{2x} \]

(E) \[ y = c_1 e^{2x} + c_2 e^{2x} \ln x \]

(F) \[ y = c_1 x^{-1} + c_2 x^{-4} \]

(G) \[ y = c_1 x + c_2 x^4 \]

(H) \[ y = c_1 x^2 + c_2 x^3 \]

(I) \[ y = c_1 x^2 + c_2 x^2 \ln x \]

(J) \[ y = c_1 x^{-2} + c_2 x^{-2} \ln x \]

4. Consider the differential equation \[ x^2 y'' - xy' + (x^2 + 1)y = 0 \]. Note that \( x_0 = 0 \) is a regular singular point. Find the form of two linearly independent series solutions about \( x_0 = 0 \).

(A) \[ y_1 = \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = \sum_{n=0}^{\infty} b_n x^n \]

\[ \lim_{x \to 0} x\left(\frac{-x}{x^2}\right) = -1 \]

(B) \[ y_1 = \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n \]

\[ \lim_{x \to 0} x^2\left(\frac{x^2 + 1}{x^2}\right) = 1 \]

(C) \[ y_1 = x \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = \sum_{n=0}^{\infty} b_n x^n \]

\[ \lambda (\lambda - 1) - \lambda + 1 = 0 \]

\[ \lambda^2 - 2\lambda + 1 = 0 \]

\( \lambda = 1, 1 \)

(D) \[ y_1 = x \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = ay_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \]

(E) \[ y_1 = x \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = x \sum_{n=0}^{\infty} b_n x^n \]

(F) \[ y_1 = x \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = y_1 \ln x + x \sum_{n=1}^{\infty} b_n x^n \]

(G) \[ y_1 = x \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = ay_1 \ln x + x \sum_{n=0}^{\infty} b_n x^n \]

(H) \[ y_1 = x^2 \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = \sum_{n=0}^{\infty} b_n x^n \]

(I) \[ y_1 = x^2 \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n \]

(J) \[ y_1 = x^2 \sum_{n=0}^{\infty} a_n x^n \]

\[ y_2 = ay_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \]
5. Consider the differential equation \( xy'' - y' + y = 0 \). Note that \( x_0 = 0 \) is a regular singular point. This differential equation has a solution of the form \( y = x^2 \sum_{n=0}^{\infty} a_n x^n \). (You do not need to verify this.) The recurrence relation for this solution is \( a_{n+1} = \frac{-1}{(n+3)(n+1)} a_n \). (You do not need to verify this either.) Let \( a_0 = 1 \), and use the recurrence relation to find the solution \( y \).

\[
\begin{align*}
(A) & \quad y = 1 - \frac{1}{3} x + \frac{1}{24} x^2 + \cdots \\
(B) & \quad y = 1 - \frac{1}{3} (x - 2) + \frac{1}{24} (x - 2)^2 + \cdots \\
(C) & \quad y = x^2 - \frac{1}{3} x^3 + \frac{1}{24} x^4 + \cdots \\
(D) & \quad y = 1 - \frac{1}{8} x + \frac{1}{120} x^2 + \cdots \\
(E) & \quad y = 1 - \frac{1}{8} (x - 2) + \frac{1}{120} (x - 2)^2 + \cdots \\
(F) & \quad y = x^2 - \frac{1}{8} x^3 + \frac{1}{120} x^4 + \cdots \\
(G) & \quad y = \frac{1}{3} - \frac{1}{8} x - \frac{1}{15} x^2 + \cdots \\
(H) & \quad y = \frac{1}{3} - \frac{1}{8} (x - 2) - \frac{1}{15} (x - 2)^2 + \cdots \\
(I) & \quad y = -\frac{1}{3} x^2 - \frac{1}{8} x^3 - \frac{1}{15} x^4 + \cdots \\

\end{align*}
\]

\( a_0 = 1 \)

\( a_1 = \frac{-1}{3} a_0 = -\frac{1}{3} \)

\( a_2 = -\frac{1}{8} a_1 = \frac{1}{24} \)

\( y = \chi^2 \sum_{n=0}^{\infty} a_n \chi^n = a_0 \chi^2 + a_1 \chi^3 + a_2 \chi^4 + \cdots \)

6. The value \( \lambda = -1 \) is an eigenvalue of the matrix \( \mathbf{A} = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \). Which of the following is an eigenvector of \( \mathbf{A} \) corresponding to \( \lambda \)?

\[
\begin{align*}
(A) & \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
(B) & \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
(C) & \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
(D) & \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
(E) & \quad \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \\
(F) & \quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \\
(G) & \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
(H) & \quad \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix} \\
(I) & \quad \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \\
(J) & \quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}
\end{align*}
\]

\[
\begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}
\]
7. The following is a phase portrait for the general solution of a certain system of differential equations. To what general solution does it correspond?

(A) \( x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} \)

(B) \( x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \)

(C) \( x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} \)

(D) \( x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \)

(E) \( x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} \)

(F) \( x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \)

(G) \( x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} \)

(H) \( x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \)

8. Consider the nonhomogeneous system of differential equations given by the following matrix equation.

\[
x' = \begin{pmatrix} 3 & 1 \\
-2 & 0
\end{pmatrix} x + \begin{pmatrix} -e^{4t} \\
-2e^{4t}
\end{pmatrix}
\]

The general solution of the corresponding homogeneous equation is as follows. (You do not need to verify this.)

\[
x_h = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{t}
\]

Begin using the method of diagonalization to solve this system. Find the function \( y_1 \) which is used in this solution method. You do not need to find \( y_2 \) or to complete the solution.

(A) \( y_1 = -4e^{-t} \)

(B) \( y_1 = -2e^{-t} \)

(C) \( y_1 = 2e^{-t} \)

(D) \( y_1 = -e^{-2t} \)

(E) \( y_1 = e^{-2t} \)

(F) \( y_1 = 3e^{-2t} \)

(G) \( y_1 = -e^{4t} \)

(\textbf{H}) \( y_1 = 2e^{4t} \)

(I) \( y_1 = 4e^{4t} \)

(J) \( y_1 = 0 \)

\[
T = \begin{pmatrix} -1 & -1 \\
1 & 2
\end{pmatrix}, \quad T^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\
t & -1
\end{pmatrix} = \begin{pmatrix} 2 & 1 \\
1 & -1
\end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix} y_1' \\
y_2'
\end{pmatrix} = \begin{pmatrix} 2 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix} y_1 \\
y_2
\end{pmatrix} + \begin{pmatrix} -2 & -1 \\
1 & 1
\end{pmatrix} \begin{pmatrix} -e^{4t} \\
-2e^{4t}
\end{pmatrix}
\]

\[
\mu(t) = e^{5-2t} = e^{-2t} e^{2t} \quad e^{-2t} y_1' = 2e^{2t} y_1, \quad y_1 = 2e^{4t}
\]

\[
\mu(t) = e^{3-2t} = e^{-2t} e^{2t} \quad y_1' = 2e^{2t} y_1, \quad y_1 = 2e^{4t}
\]
9. Consider the following partial differential equation.

\[ tu_{xx} - u_t = 0 \]

Which one of the following pairs of ordinary differential equations can be obtained from this partial differential equation using the method of separation of variables?

(A) \[ X'' - \lambda X = 0 \quad T' - \lambda t T = 0 \]
(B) \[ X'' - \lambda X = 0 \quad t T' - \lambda T = 0 \]
(C) \[ X' - \lambda X = 0 \quad T'' - \lambda t T = 0 \]
(D) \[ X' - \lambda X = 0 \quad t T' - \lambda T = 0 \]
(E) \[ X'' - \lambda = 0 \quad T' - \lambda t = 0 \]
(F) \[ X'' - \lambda = 0 \quad t T' - \lambda = 0 \]
(G) \[ t X'' - \lambda = 0 \quad T' - \lambda = 0 \]
(H) \[ X'' - \lambda t = 0 \quad T' - \lambda = 0 \]
(I) The variables cannot be separated.

10. Simplify the following Fourier series.

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2} \left[ 1 - \cos(-n\pi) \right] \cos n\pi x \]

(A) 0
(B) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \pi^2} \cos n\pi x \]
(C) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 \pi^2} \cos n\pi x \]
(D) \[ \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2 \pi^2} \cos n\pi x \]

(E) \[ \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^2 \pi^2} \cos n\pi x \]
(F) \[ \sum_{k=1}^{\infty} \frac{-2}{(2k-1)^2 \pi^2} \cos(2k-1)\pi x \]

(G) \[ \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 \pi^2} \cos(2k-1)\pi x \]
(H) \[ \sum_{k=1}^{\infty} \frac{2}{(2k-1)^2 \pi^2} \cos(2k-1)\pi x \]
(I) \[ \sum_{k=1}^{\infty} \frac{2}{(2k-1)^2 \pi^2} \cos(2k-1)\pi x \]
Part II. True-False (1 point each)

Mark your answer card "A" if the statement is true and "B" if the statement is false.

11. The functions $f_1(t) = t^2$ and $f_2(t) = t^{-2}$ are linearly independent.
   
   $A$

12. Power series methods are especially useful for solving differential equations with non-constant coefficients.
   
   $A$

13. The Laplace transform method is especially useful for solving differential equations with discontinuous forcing functions.
   
   $A$

14. Every square matrix which is Hermitian is also diagonalizable.
   
   $A$

15. Every boundary value problem is guaranteed to have a unique solution.
   
   $B$
Part III. Computational

16. (3 points)

Two tanks are connected as shown in the diagram below. The first tank contains 100 gal of salt water, and the second tank contains 200 gal of salt water. Pure water flows into the first tank at the rate of 4 gal/min. The well-stirred mixture in the first tank flows into the second tank at the rate of 4 gal/min. Salt water containing 5 lb of salt per gallon flows into the second tank from another source at the rate of 2 gal/min. The well-stirred mixture in the second tank flows out at the rate of 6 gal/min. (Note that the rate of inflow equals the rate of outflow for each tank, so the volume of salt water in each does not change.)

Let \( x(t) \) represent the quantity of salt in the first tank and let \( y(t) \) represent the quantity of salt in the second tank at time \( t \). (Note that both \( x \) and \( y \) are measured in lbs.) Your task is to write the differential equation which represents the rate at which the quantity of salt in the second tank is changing. In other words, finish the following differential equation.

\[
\frac{dy}{dt} = \left(4 \left( \frac{x}{100} \right) \right) + \left(2 \times 5 \right) - \left(6 \left( \frac{y}{200} \right) \right)
\]

(Hints: Do not panic! Although you have not seen this tank set-up before, the principles involved in setting up the differential equation are exactly the same as for the set-ups you have seen. Note that the differential equation can involve any or all of the variables \( x, y, \) and \( t \). You do NOT need to simplify or solve your equation.)
17. (3 points)

Consider the following sixth order linear homogeneous differential equation.

\[ y^{(6)} - 3y''' - 2y'' + 22y'' + 33y' + 13y' = 0 \]

The characteristic equation for this differential equation has the following roots. (You do not need to verify this.)

\[ 0, -1, -1, -1, 3\pm2i \]

Write the general solution of the differential equation.

\[ y = c_1 + c_2 e^{-t} + c_3 t e^{-t} + c_4 t^2 e^{-t} + c_5 e^{3t} \cos 2t + c_6 e^{3t} \sin 2t \]
18. (8 points)

Use Laplace transforms to solve the following initial value problem. Show your work. You do not need to simplify any trigonometric functions in your final answer.

\[ y'' + 2y' + 5y = 10u_3(t), \ y(0) = 0, \ y'(0) = 0 \]

\[ s^2Y(s) + 2sY(s) + 5Y(s) = \frac{10e^{-3s}}{s} \]

\[ Y(s) = e^{-3s} \left[ \frac{10}{s(s^2 + 2s + 5)} \right] \]

\[ \frac{10}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5} \]

\[ 10 = A(s^2 + 2s + 5) + (Bs + C)(s) \]

\[ 0 = A + B \]

\[ 0 = 2A + C \]

\[ 10 = 5A \]

\[ A = 2 \quad B = -2 \quad C = -4 \]

\[ \frac{10}{s(s^2 + 2s + 5)} = \frac{2}{s} + \frac{-2s - 4}{(s^2 + 2s + 5)} = \frac{2}{s} + \frac{-2s - 4}{(s+1)^2 + 4} \]

\[ = \frac{2}{s} + \frac{D(s+1)}{(s+1)^2 + 4} + \frac{E(2)}{(s+1)^2 + 4} \]

\[ D = -2 \quad E = -1 \]

\[ \mathcal{L}^{-1} \left\{ \frac{10}{s(s^2 + 2s + 5)} \right\} = 2 - 2e^{-t}\cos 2t - e^{-t}\sin 2t \]

\[ y(t) = \mathcal{L}^{-1} \{ Y(s) \} = \mathcal{L}^{-1} \left[ e^{-3s} \left[ \frac{10}{s(s^2 + 2s + 5)} \right] \right] \]

\[ = u_3(t) \left[ 2 - 2e^{-(t-3)}\cos 2(t-3) - e^{-(t-3)}\sin 2(t-3) \right] \]
19. (5 points)

Consider the following system of equations.

\[ x' = \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \end{pmatrix} \]

The eigenvalues and corresponding eigenvectors for the matrix are as follows. (You do not need to verify this.)

\[ \lambda_1 = 2i \quad \xi^{(1)} = \begin{pmatrix} 1 + 2i \\ -1 \end{pmatrix} \]
\[ \lambda_2 = -2i \quad \xi^{(2)} = \begin{pmatrix} 1 - 2i \\ -1 \end{pmatrix} \]

Find the general solution of this system, expressed in terms of real-valued functions. Show your work.

\[ x^{(1)}(t) = e^{2it} \begin{pmatrix} 1 + 2i \\ -1 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 + 2i \\ -1 \end{pmatrix} \left( \cos 2t + i \sin 2t \right) \]
\[ = \begin{pmatrix} \cos 2t + i \sin 2t + 2i \cos 2t - 2 \sin 2t \\ -\cos 2t - i \sin 2t \end{pmatrix} \]
\[ = \begin{pmatrix} \cos 2t - 2 \sin 2t \\ -\cos 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t + 2 \cos 2t \\ -\sin 2t \end{pmatrix} \]

\[ u(t) = \begin{pmatrix} \cos 2t - 2 \sin 2t \\ -\cos 2t \end{pmatrix} \quad \nu(t) = \begin{pmatrix} \sin 2t + 2 \cos 2t \\ -\sin 2t \end{pmatrix} \]

\[ x(t) = c_1 \begin{pmatrix} \cos 2t - 2 \sin 2t \\ -\cos 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t + 2 \cos 2t \\ -\sin 2t \end{pmatrix} \]
20. (6 points)

Find the solution of the following heat conduction problem. Show your work.

\[4u_{xx} = u_t, \quad u(0, t) = 0, \quad u(\pi, t) = 0, \quad u(x, 0) = 20\]

You do NOT need to simplify your answer (as you did in problem 10).

\[L^2 = 4, \quad L = \pi\]

\[u(x, t) = \sum_{n=1}^{\infty} c_n e^{-4n^2 t} \sin nx\]

where \(c_n = \frac{2}{\pi} \int_0^\pi 20 \sin nx \, dx\)

\[= \frac{40}{\pi} \left(\frac{-1}{n}\right) \cos nx \bigg|_0^\pi\]

\[= \frac{-40}{\pi n} (\cos n\pi - 1)\]

\[u(x, t) = \sum_{n=1}^{\infty} \frac{-40}{\pi n} (\cos n\pi - 1) e^{-4n^2 t} \sin nx\]