Course: Math 233

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- To see your exam score, go the the math department homepage at www.math.wustl.edu and use the link to 'Exam Scores' under 'Resources'.

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EXAM 1, MATH 233
FALL, 2003

This examination has 20 multiple choice questions, and two essay questions. Please check it over and if you find it to be incomplete, notify the proctor. Do all your supporting calculations in this booklet. In case of a doubtful mark on your answer card, your instructor can then check here. When you mark your card, use a soft lead pencil (#2). Erase fully any answers you want to change. Problems 1 through 20 are worth 3.5 points apiece for a total of 70 points.

On problems 21 and 22, show all your work and indicate clearly your answer to the problem. Partial credit will be given for partially completed solutions on these two problems. Each of these problems is worth 15 points for a total of 30 points.

There is a total of 100 points for the whole examination.
You may use a scientific calculator and a 3 x 5 note card.

(1) Find the radius of the sphere with equation \( x^2 + y^2 + z^2 = x + y + z. \)
- (A) \( 1/2; \)
- (B) \( 1; \)
- (C) \( 2; \)
- (D) \( 5/2; \)
- (E) \( \sqrt{2}; \)
- (F) \( \sqrt{3}/2; \)
- (G) \( \sqrt{3}; \)
- (H) \( \sqrt{5}; \)
- (I) \( \sqrt{5}/2; \)
- (J) \( \sqrt{6}; \)

\[
\begin{align*}
(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + (z - \frac{1}{2})^2 &= \frac{3}{4} \\
\sqrt{r} &= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}
\end{align*}
\]
(2) The perpendicular distance from the point \((3, 7, -5)\) to the x-axis is

- (A) 0
- (B) \(74^{1/2}\)
- (C) 3.5
- (D) -3.5
- (E) 77\(\pi\)
- (F) 10
- (G) 3
- (H) 7
- (I) 5
- (J) 83\(^{1/2}\)

The point on the x-axis closest to \(P(3, 7, -5)\) is \(Q(3, 0, 0)\) with

\[|PQ| = \sqrt{7^2 + 5^2} = \sqrt{74} = 74^{1/2}\]

(3) The line that passes through the point \((0, 2, -1)\) and is parallel to the line

\[r = \langle 1, 0, 5 \rangle + t < 2, 3, -7 >\] has equation:

- (A) \(x = 2t, y = 2 + 3t, z = -1 - 7t;\)
- (B) \(x = 1, y = 2t, z = 5 - t;\)
- (C) \(x = 3, \frac{y - 2}{3} = \frac{z + 7}{5};\)
- (D) \(x = 3, \frac{y - 2}{3} = \frac{z + 7}{5};\)
- (E) \(x = 3, \frac{y - 2}{3} = \frac{z + 7}{5};\)
- (F) \(x = 3, \frac{y - 2}{3} = \frac{z + 7}{5};\)
- (G) \(r = (i - 2j + 6k) + t(2i + 3j - 7k);\)
- (H) \(r = (10i + j - 2k) + t(2i + 3j - 7k);\)
- (I) \(2x + 3(y - 2) - 7(z + 1) = 0;\)
- (J) \(2(y - 3) - (z + 7) = 0.\)

\(\vec{v} = <2, 3, -7>\) is a direction vector for both lines. Parametric equations for the desired line are

\[
\begin{align*}
x &= 0 + 2t \\
y &= 2 + 3t \\
z &= -1 - 7t
\end{align*}
\]
(4) The direction numbers of the line $x = 4 + 3t, y = 5 - 2t$ and $z = 2 + t$ are

(A) $(4, -2, 1)$  
(B) $(3, 5, 2)$  
(C) $(3, 4, 1)$  
(D) $(5, -2, 0)$  
(E) $(2, 1, 0)$  
(F) $(0, 0, 0)$  
(G) $(3, -6, 7)$  
(H) $(1, 3, 5)$  
(I) $(2, -3, 4)$  

(J) $(3, -2, 1)$  

\[\sqrt{3} = \langle 3, -2, 1 \rangle\] is a direction vector for the line with $(3, -2, 1)$ direction numbers.

(5) Let $a = i - j, b = i + k$. Then the vector projection of $b$ onto $a$ is:

\[\text{proj}_a b = \left( \frac{\langle b \cdot a \rangle}{a \cdot a} \right) a\]

\[= \left( \frac{\langle i, -1, 0 \rangle \cdot \langle 1, 0, 1 \rangle}{\langle 1, -1, 0 \rangle \cdot \langle 1, -1, 0 \rangle} \right) (i - j)\]

\[= \frac{1}{2} (i - j) - \frac{1}{2} j\]
(6) Let \( \mathbf{a} = -2i + j + 3k \), \( \mathbf{b} = 3i + 3j + k \). Then the angle between \( \mathbf{a} \) and \( \mathbf{b} \) is:

\[
\left| \mathbf{a} \right| \left| \mathbf{b} \right| \cos \theta = \mathbf{a} \cdot \mathbf{b} \\
= \langle -2, 1, 3 \rangle \cdot \langle 3, 3, 1 \rangle \\
= -6 + 3 + 3 \\
= 0 \\
\]

So \( \theta = \pi/2 \)

(7) The equations of two lines are as follows:

\[
x = t - 5, \quad y = 3t + 1, \quad z = -0.5t + 2
\]

\[
\frac{x}{2} = \frac{y}{6} = \frac{z + 1}{-1}
\]

Then the two lines:
(A) are skew lines;
(B) perpendicular to each other;
(C) are parallel;
(D) intersect at an angle of \( \pi/4 \);
(E) intersect at an angle of \( \pi/3 \);
(F) intersect at an angle of \( \pi/6 \);
(G) coincide;
(H) intersect at an angle of \( 2\pi/3 \);
(I) intersect at an angle of \( 3\pi/4 \);
(J) intersect at an angle of \( 5\pi/6 \);

Direction vectors for the lines are:
\( \mathbf{v}_1 = \langle 1, 3, -\frac{1}{2} \rangle \)
and \( \mathbf{v}_2 = \langle 2, 6, -1 \rangle \)

Since \( \mathbf{v}_2 = 2 \mathbf{v}_1 \), the lines are parallel.
(8) Find an equation for the line of intersection of the two planes \( x - y + z = 2 \) and \( 2x + y + 3z = 0 \).

(A) \( x = -2t + 2, y = -t - 4, z = 3t; \)
(B) \( x = -t + 3, y = -2t - 3, z = t; \)
(C) \( x = -4t, y = -3t - 1, z = 2t; \)
(D) \( x = -4t + 2/3, y = -t - 4/3, z = 3t; \)
(E) \( x = -t + 2/3, y = -2t - 4/3, z = t; \)
(F) \( x = -t + 2/3, y = -t - 4/3, z = t; \)
(G) \( x = 2t + 2/3, y = t - 4/3, z = 3t; \)
(H) \( x = -3t + 2/3, y = t - 4/3, z = 2t; \)
(I) \( x = 3t + 2/3, y = -t - 4/3, z = -2t; \)
(J) \( x = -6t + 2/3, y = -2t - 4/3, z = 4t; \)

Normal vectors for the two planes are
\( \vec{n}_1 = \langle 1, -1, 1 \rangle \)
and \( \vec{n}_2 = \langle 2, 1, 3 \rangle \)
with \( \vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle -4, -1, 3 \rangle \)
a direction vector for their line of intersection. Putting \( t = 0 \),

the equations \( x - y = 2 \)
\( 2x + y = 0 \)

have the solution \( x = 2/3, y = -4/3 \).

So \( P_0 \left( 3/3, -4/3, 0 \right) \)
is a point on the line of intersection and this line has

equations
\( x = \frac{3}{3} - 4t \)
\( y = -4/3 + t \)
\( z = 0 + 3t \)

(9) The plane passing through \( P(1, 1, 1) \) and containing the line \( x = 1 + t, y = -2 + 3t, z = 4 - t \) has the equation:

(A) \( 3x - 2y + z - 2 = 0; \)
(B) \( x - 2y + z - 2 = 0; \)
(C) \( 2x - 4y + z = 0; \)
(D) \( -x - 2y + z - 3 = 0; \)
(E) \( 4x - y + z - 6 = 0; \)
(F) \( 3x - 2y + 4z - 7 = 0; \)
(G) \( 2x - y - z = 0; \)
(H) \( x - 2y + 3z - 4 = 0; \)
(I) \( -2x - 6y + z + 5 = 0; \)
(J) \( 6x - 2y + 3z + 12 = 0; \)

Taking \( t = 1 \),

\( Q(1, -2, 4) \) is a point on the given line \( L \)
and \( \vec{v} = \langle 1, 3, -1 \rangle \) is the direction vector for \( L \). As a normal vector for the plane, we can take either \( \vec{v} \times \vec{n} \)
\( = \langle 0, -3, 3 \rangle \)
\( x \langle 1, 3, -1 \rangle \)
\( = \langle -3, 3, 3 \rangle \)
\( \vec{n} = \frac{1}{3} \langle -3, 3, 3 \rangle = \langle -2, 1, 1 \rangle \)

The plane has the equation
\( -2x + y + z = \langle x, y, z \rangle \cdot \vec{n} = \langle 1, 1, 1 \rangle \cdot \vec{n} = 0 \)

or \( \langle x - y - z = 0 \rangle \)
(10) The distance between the two skew lines \( x = 1 + t, y = -2 + 3t, z = 4 - t \) and 
\( x = 2s, y = s, z = s \) is:
(A) \( \frac{1}{2} \);
(B) \( \frac{3}{4} \);
(C) \( 1 \);
(D) \( \frac{5}{4} \);
(E) \( \sqrt{2} \);
(F) \( 1.5 \);
(G) \( \sqrt{3} \);
(H) \( 2 \);
(I) \( 2.5 \);
(J) \( 4 \);

Direction vectors for the lines are
\[ \vec{v}_1 = <1, 3, -1> \]
and \[ \vec{v}_2 = <2, 1, 1> \]

with \( \vec{n} = \vec{v}_1 \times \vec{v}_2 = <4, -3, -5> \) perpendicular to both lines. Then the planes

\[ 4x - 3y - 5z = \vec{n} \cdot <1, -2, 4> = -10 \]
and \[ 4x - 3y - 5z = \vec{n} \cdot <0, 0, 6> = 0 \]

are parallel and the distance \( \frac{10}{\sqrt{50}} = \frac{10}{5} \sqrt{2} = \frac{2}{\sqrt{2}} = \sqrt{2} \)
between the planes is also the distance between the two lines.

(11) The area of the parallelogram with vertices \( A(2, 1, 1), B(4, 1, 3), C(3, -2, 1), \) and 
\( D(5, -2, 3) \) is:
(A) \( 5 \);
(B) \( 10 \);
(C) \( 12 \);
(D) \( 18 \);
(E) \( 25 \);
(F) \( \sqrt{15} \);
(G) \( \sqrt{42} \);
(H) \( \sqrt{67} \);
(I) \( \sqrt{76} \);
(J) \( \sqrt{91} \);

\[ \overrightarrow{AB} = \overrightarrow{CD} = <2, 0, 2> \]
\[ \overrightarrow{AC} = \overrightarrow{BD} = <1, -3, 0> \]

Area \( \square ABCD = | \overrightarrow{AB} \times \overrightarrow{AC} | \)
\[ = | <6, 2, -6> | \]
\[ = \sqrt{36 + 4 + 36} \]
\[ = \sqrt{76} \]
(12) The point of intersection of the line \( x = 2t - 5, y = 3t + 1, z = -t + 2 \) with the plane \( x + y + z = 10 \) is:
(A) \((-5, 1, 2)\);
(B) \((-7, -2, -3)\);
(C) \((-3, 4, 1)\);
(D) \((-1, 7, 0)\);
(E) \((1, 10, -1)\);
(F) \((3, 13, -2)\);
(G) \((2, 3, 5)\);
(H) \((6, 1, 3)\);
(I) \((3, 3, 4)\);
(J) \((2, 4, 4)\);

\[
\begin{align*}
(2t - 5) + (3t + 1) + (-t + 2) &= 10 \\
4t &= 10 \\
t &= 3 \\
x &= 2 \cdot 3 - 5 = 1 \\
y &= 3 \cdot 3 + 1 = 10 \\
z &= -3 + 2 = -1 \\
P(1, 10, -1) & \text{ is the point of intersection}
\end{align*}
\]

(13) Given \( A(1, 2, 3) \) and \( B(-1, 3, 0) \). Then the set of points \( P \) equidistant from \( A \) and \( B \) is the plane whose equation is:
(A) \( 2x - y + 3z - 2 = 0 \);
(B) \( 3x - y + 2z - 4 = 0 \);
(C) \( -2x - 6y + 3z = 0 \);
(D) \( -x - 2y + 3z - 3 = 0 \);
(E) \( 4x - y + 3z - 6 = 0 \);
(F) \( x - 3y - z - 1 = 0 \);
(G) \( x - 3y + 3z + 1 = 0 \);
(H) \( -4x - y + 3z - 5 = 0 \);
(I) \( 6x - 7y + 4z - 8 = 0 \);
(J) \( 7x + 8y + z - 9 = 0 \);

\[
\begin{align*}
\vec{A} \vec{B} &= \begin{pmatrix} 2 \end{pmatrix} \\
\hat{n} &= \text{a normal vector for the plane } M = \text{midpoint of } AB \\
m(0, \frac{5}{2}, \frac{3}{2}) &= m, A(1, 2, 3), B(-1, 3, 0) \\
\text{a point on the plane} \\
\text{so the plane's equation is} \\
\begin{pmatrix} x, y, z \end{pmatrix} \cdot \hat{n} &= 2x - y + 3z = \begin{pmatrix} 0, \frac{5}{2}, \frac{3}{2} \end{pmatrix} \cdot \hat{n} \\
&= -\frac{5}{2} + \frac{9}{2} \\
&= 2
\end{align*}
\]
(14) For what pair of values \(k, l\) does the point \(R(k, l, 0)\) lies on the line through the points \(P(2, -1, 3)\) and \(Q(2, 3, -1)\)?

(A) 5,3;  
(B) 4,4;  
(C) 3,1;  
(D) 1,-1;  
(E) 0,0;  
(F) 2,2;  
(G) -1,-3;  
(H) -2,-2;  
(I) -3,-5;  
(J) -4,-4;  

The vector equation of the line is

\[
\langle x, y, z \rangle = \overrightarrow{OP} + t \overrightarrow{PQ} \\
= \langle 2, -1, 3 \rangle + t \langle 0, 4, -4 \rangle
\]

Then \(3 = 0\) for \(t = \frac{3}{4}\)

with \(x = 2 + 0 = 2\)

\(y = -1 + \frac{3}{4}(4) = 2\)

so \(R(2, 2, 0)\) is the desired point.

(15) Find the perpendicular distance between the planes with equations \(x - 2y + 2z = 4\) and \(2x - 4y + 4z = 6\).

(A) 1/6;  
(B) 1/4;  
(C) 1/3;  
(D) 1/2;  
(E) 1;  
(F) \(\sqrt{1.5}\);  
(G) \(1/\sqrt{3}\);  
(H) \(3/\sqrt{2}\);  
(I) \(\sqrt{5/3}\);  
(J) \(\sqrt{2/3}\);

We can rewrite the equations as

\[
\langle x, y, z \rangle \cdot \langle 1, -2, 2 \rangle = 2 = d_1, \\
\langle x, y, z \rangle \cdot \langle 1, -2, 2 \rangle = 3 = d_2
\]

So \(\vec{n} = \langle 1, -2, 2 \rangle\) is a common normal vector and

\[
\left| \frac{d_1 - d_2}{|\vec{n}|} \right| \text{ is the perpendicular distance between the planes.}
\]
(16) Suppose the force $\mathbf{F} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ moves a particle from $(1, 1, 1)$ to $(2, 2, 2)$. The work done is:

\[
\text{Work done} = \mathbf{F} \cdot \mathbf{AB} = \langle -2, 1, 3 \rangle \cdot \langle 1, 1, 1 \rangle = -2 + 1 + 3 = 2
\]

\[\text{Answer: (G) 2.}\]

(17) Find the output $d$ of the following MATLAB script: $a = [2, -3, 4]$

\[
b = [1, 2, 3];
\]

\[
d = \text{sum}(a \cdot b) = \langle 2, -3, 4 \rangle \cdot \langle 1, 2, 3 \rangle
\]

\[= 2 - 6 + 12 = 8
\]

\[\text{Answer: (D) 8.}\]
(18) For all non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, the vectors $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ are

(A) orthogonal to each other;
(B) parallel to each other;
(C) make an angle of $\pi/4$ with each other;
(D) make an angle of $\pi/6$ with each other;
(E) make an angle of $\pi/8$ with each other;
(F) make an angle of $\pi/10$ with each other;
(G) the same vector;
(H) positive multiples of each other;
(I) diagonals of a parallelogram with positive area;
(J) both the zero vector.

\[
\mathbf{b} \times \mathbf{a} = - \left( \mathbf{a} \times \mathbf{b} \right)
\]

is a multiple of $\mathbf{a} \times \mathbf{b}$

so the two vectors are parallel to each other.

(19) Which of the following equations describes a surface whose horizontal traces are hyperbolas and whose vertical traces are parabolas?

(A) $2x - y + 3z - 2 = 0$;
(B) $3x^2 - y^2 + 2z^2 = 4$;
(C) $2x^2 + 6y^2 + 3z^2 = 17$;
(D) $x^2 - 2y^2 - 3z = 0$;
(E) $4x^2 - y + -6 = 0$;
(F) $x^2 = 3y^2 + z^2$;
(G) $2x^2 - 3y^2 = 1$;
(H) $-4x - y^2 + 3z = 5$;
(I) $6x^2 + 7y^2 - 4z = 0$;
(J) $x^2 + 8y^2 + 3 = z^2$;

(D) is the equation of a hyperbolic paraboloid, i.e. a quadric surface with horizontal traces hyperbolas and vertical traces parabolas.
(20) Consider the quadric surface with equation \( z^2 = 2x^2 + 3y^2 + 4 \). Which of the following pairs describes the types of curves arising as horizontal and vertical traces for this surface?

(A) ellipses and ellipses;
(B) parabolas and ellipses;
(C) hyperbolas and parabolas;
(D) ellipses and parabolas;
(E) hyperbolas and straight lines;
(F) circles and parabolas;
(G) ellipses and straight lines;
(H) ellipses and hyperbolas;
(I) parabolas and straight lines;
(J) straight lines and straight lines.

Horizontal trace:
\[ 2x^2 + 3y^2 = \text{constant} \]
(Ellipses)

Vertical trace:
Either \( z^2 - 2x^2 = \text{constant} \) or \( z^2 - 3y^2 = \text{constant} \)
(Hyperbolas)

The surface is a hyperboloid with 2 sheets.
(21) Write your answers to parts (a)-(c) in the space provided and, if you need additional space, on the back of this page. Try to write legibly and neatly. If an answer from one part is needed to answer another part and you’re unable to obtain the first answer, state the method you would use to answer the second part if you had the first answer.

Given 4 points \( P(1, 0, 1) \), \( Q(3, -1, 2) \), \( R(2, 4, 6) \) and \( S(1, 1, 1) \), do the following:

(a) Find an equation for the plane containing \( P, Q, \) and \( R \).

\[
\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 4 & 5 \\
2 & -1 & 1
\end{vmatrix} = \begin{pmatrix} 9 \\ 9 \\ -9 \end{pmatrix}
\]

\( \vec{n} \) is a normal vector for the plane and so \( \vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \).

Equation of the plane:
\[
x + y - z = \langle x, y, z \rangle \cdot \vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0
\]

(b) Find the perpendicular distance from \( S \) to the plane in (a);

\[
distance \text{ from } S \text{ to the plane } = \frac{|1 + 1 - 1 - 0|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{1}{\sqrt{3}}
\]

(c) Find an equation for the sphere with center \( S \) that touches the plane containing \( P, Q, R \) in exactly one point.

The radius of the sphere is \( |S| = 1 = \frac{1}{\sqrt{3}} \) (from (b)).

Since the foot \( F \) of the perpendicular dropped from \( S \) to the plane must be the unique point where the sphere touches the plane.

Then \( (x - 1)^2 + (y - 1)^2 + (z - 1)^2 = \frac{1}{3} \) is the equation of the sphere.
(22) Follow the same directions as in the previous problem. Let \( \mathbf{a} = \langle 1, 2, 2 \rangle \) and \( \mathbf{b} = \langle 4, 5, 2 \rangle \):

(a) Find vectors \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) for which \( \mathbf{b}_1 \) is a scalar multiple of \( \mathbf{a} \), \( \mathbf{b}_2 \) is orthogonal to \( \mathbf{a} \), and \( \mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2 \):

\[
\mathbf{b}_1 = \text{Proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a} = \frac{4 + 10 + 4}{1 + 4 + 4} \mathbf{a} = \frac{18}{9} \mathbf{a} = 2 \mathbf{a} = \langle 2, 4, 4 \rangle
\]

\[
\mathbf{b}_2 = \mathbf{b} - \mathbf{b}_1 = \langle 4, 5, 2 \rangle - \langle 2, 4, 4 \rangle = \langle 2, 1, -2 \rangle
\]

(b) Find a unit vector \( \mathbf{u} \) in the direction of \( \mathbf{a} \) and a unit vector \( \mathbf{v} \) in the direction of \( \mathbf{b}_2 \).

\[
\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \langle 1, 2, 2 \rangle
\]

\[
\mathbf{v} = \frac{\mathbf{b}_2}{\|\mathbf{b}_2\|} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \langle 2, 1, -2 \rangle
\]

(c) Use (a) and (b) to express \( \mathbf{a} \) and \( \mathbf{b} \) as combinations of \( \mathbf{u} \) and \( \mathbf{v} \).

Since \( \mathbf{a} = 3 \mathbf{u} \) and \( \mathbf{b}_2 = 3 \mathbf{v} \):

\[
\mathbf{b} = \frac{2 \mathbf{a}}{2} + \mathbf{b}_2 = \frac{2}{3} \begin{bmatrix} 1 \end{bmatrix} \langle 1, 2, 2 \rangle + \mathbf{b}_2 = \begin{bmatrix} 6 \end{bmatrix} \mathbf{u} + 3 \mathbf{v}
\]