Solutions

EXAM 3, MATH 233

WEDNESDAY, NOVEMBER 13, 2002

This examination has 20 multiple choice questions and two essay questions. Please check over your test booklet at the outset and if you find it to be incomplete, notify one of the proctors. Do all of your supporting calculations in this booklet. In case of a doubtful mark on your answer card, your instructor can then review your work and possibly give you credit for the problem in question. When you mark your card, use a a soft lead pencil (#2). Erase fully any answers you decide to change. Problems 1 through 20 are worth 3.5 points each.

Problems 21 and 22 are essay questions worth fifteen points each. **Be sure to write your name and Student ID# on both of the essay question pages; in the grading process, these pages will be separated from the rest of the test booklet. Show all of your work on the two essay questions and clearly identify your answers.** Partially completed solutions will be given partial credit. However, when a problem involves a calculation and no work is shown, no credit will be given for simply recording a correct answer.

The maximum score for the entire examination is 100 points. You may use a graphing calculator and may refer to a 3 x 5 note card.

1. Find the maximum value of $f(x, y) = xy$ for points $(x, y)$ satisfying $2x + y = 12$

   (A) 12  (F) 21
   (B) 15  (G) 24
   (C) 16  (H) 26
   (D) 18   (I) 30
   (E) 20   (J) 36

   $\nabla f = \begin{bmatrix} \nabla (xy) \end{bmatrix}$

   $\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 2, 1 \end{bmatrix}$

   $\frac{y}{2} = \frac{2}{2} = \frac{x}{1}$

   $y = 2x$

   $12 = 2x + y = 4x$

   $x = 3, \ y = 6$ and $f(3, 6) = 18$
2. Suppose the temperature in degrees Celsius at a point \((x, y)\) on a metal plate is given by the formula

\[
T(x, y) = 100e^{-(3x^2+y^2)}.
\]

Which of the following unit vectors gives the direction in which \(T\) decreases most rapidly at the point \((1,4)\)?

(A) \(< -1, -1 > / \sqrt{2}\)  
(B) \(< 1, 4 > / \sqrt{17}\)  
(C) \(< -3, -1 > / \sqrt{10}\)  
(D) \(< 1, 0 >\)  
(E) \(< 0, 1 >\)  
(F) \(< 3, 4 > / 5\)  
(G) \(< 2, 3 > / \sqrt{13}\)  
(H) \(< -4, -3 > / 5\)  
(I) \(< 1, 1 > / \sqrt{2}\)  
(J) \(< 5, 2 > / \sqrt{29}\)

\[
\text{Direction} = -\frac{\nabla T(1,4)}{|\nabla T(1,4)|} = 100 \frac{\langle 6, 8 \rangle}{100 \langle 6, 8 \rangle} = \frac{\langle 3, 4 \rangle}{\langle 3, 4 \rangle}
\]

3. The points \(P_0(0,0)\) and \(P_1(1,-1)\) are two of the four critical points of the function

\[
f(x, y) = 3xy^2 + x^3 - 3y^2 - 3x^2 + 5
\]

Which of the following describes the behaviour of \(f\) at \(P_0\) and \(P_1\)?

(A) local maximum at \(P_0\) and local minimum at \(P_1\)  
(B) saddle point at \(P_0\) and local minimum at \(P_1\)  
(C) local maximum at both \(P_0\) and \(P_1\)  
(D) local minimum at both \(P_0\) and \(P_1\)  
(E) saddle point at both \(P_0\) and \(P_1\)  
(F) local minimum at \(P_0\) and local maximum at \(P_1\)  
(G) saddle point at \(P_0\) and local maximum at \(P_1\)  
(H) local maximum at \(P_0\) and saddle point at \(P_1\)  
(I) local minimum at \(P_0\) and behaviour at \(P_1\) undetermined  
(J) behaviour at \(P_0\) undetermined and local minimum at \(P_1\)

At \((0,0)\), 
\[
D = f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{so}
\]

\((0,0)\) is a local maximum point.

At \((1,-1)\), 
\[
f_{xx} = 0 = f_{yy} \quad \text{and}
\]

\[D = -36 < 0 \quad \text{so}
\]

\((1,-1)\) is a saddle point.
4. Find the critical points of the function \( f(x, y) = y^3 + 3xy + 3x^2/2. \)

(A) \((-1, 1)\) and \((1, -1)\)  
(B) \((0, 0)\) and \((1, 3)\)  
(C) \((-1, -1)\) and \((0, 0)\)  
(D) \((2, 1)\) and \((-1, -1)\)  
(E) \((2, 2)\) and \((-1, -1)\)  
(F) \((1, 3)\) and \((3, 1)\)  
(G) \((-2, -2)\) and \((1, 1)\)  
(H) \((-1, 1)\) and \((2, 2)\)  
(I) \((0, 0)\) and \((-1, 1)\)  
(J) \((1, -3)\) and \((3, -1)\)

\[
\begin{align*}
f_x &= 3y + 3x = 3(y + x) \\
f_y &= 3y^2 + 3x = 3(y^2 + x) \\
0 &= f_x = f_y \quad \Rightarrow \quad y = -x = y^2 \\ & \text{so either} \\
(x, y) &= (0, 0) \quad \text{or} \quad (x, y) = (-1, 0)
\end{align*}
\]

5. Find the minimum value of the function \( f(x, y) = x^2 + y^2 \) subject to the constraint \( x^4 + y^4 = 1. \)

(A) \(2\)  
(B) \(\sqrt{3}\)  
(C) \(3/2\)  
(D) \((\sqrt{5})/2\)  
(E) \(1 + \sqrt{2}/2\)  
(F) \(\sqrt{2}\)  
(G) \(4/3\)  
(H) \(5/4\)  
(I) \(\sqrt{7}/3\)  
(J) \(1\)  

\[
\begin{align*}
\nabla f &= \nabla (x^4, y^4) \\
\langle 2x, y \rangle &= 4 \langle x^3, y^3 \rangle \\
\text{when} \quad x = 0, \quad y^4 = 1 \quad \text{and} \quad f = 1 \quad \boxed{f = 1} \\
\text{when} \quad y = 0, \quad \boxed{f = 1} \\
\text{when} \quad x \text{ and } y \text{ are non-zero} \\
\frac{x}{x^3} &= 2x = \frac{y}{y^3} \\
\Rightarrow \quad y^2 &= x^2 = \frac{1}{\sqrt{2}} \\
\text{and} \quad \nabla f &= \frac{2}{\sqrt{2}} = \sqrt{2}
\end{align*}
\]

Hence, 1 must be the minimum value.
6. Calculate $\int \int_R xe^{xy} \, dA$ for $R = [0, 1] \times [0, 1]$.

\[
\int \int_R xe^{xy} \, dA = \int_0^1 \int_0^1 xe^{xy} \, dy \, dx = \int_0^1 e^{xy} \Big|_{y=0}^{y=1} \, dx = \int_0^1 (e^x - 1) \, dx = e - 2
\]

7. Which of the following is equal to $\int_0^{16} \int_0^{x^{1/4}} f(x, y) dy \, dx$ for any integrable function $f$?

(A) $\int_0^{16} \int_0^{x^{1/4}} f(x, y) dx \, dy$

(B) $\int_0^{16} \int_0^{y^4} f(x, y) dx \, dy$

(C) $\int_0^{16} \int_0^{y^4} f(x, y) dx \, dy$

(D) $\int_0^{16} \int_0^{y^4} f(x, y) dx \, dy$

(E) $\int_0^{16} \int_0^{y^4} f(x, y) dx \, dy$

(F) $\int_0^{16} \int_0^{x^{1/4}} f(x, y) dx \, dy$

(G) $\int_0^{16} \int_0^{y^4} f(x, y) dx \, dy$

(H) $\int_0^{16} \int_0^{y^4} f(x, y) dx \, dy$

(I) $\int_0^{16} \int_0^{y^4} f(x, y) dx \, dy$

(J) $\int_0^{16} \int_0^{y^4} f(x, y) dx \, dy$
8. Find the volume of the solid region lying below the surface \( z = x^2y \) and above the rectangle \([0, 1] \times [1, 4]\) in the \(xy\) plane.

(A) \(\frac{1}{4}\)  
(B) \(\frac{1}{2}\)  
(C) \(\frac{3}{4}\)  
(D) \(1\)  
(E) \(\frac{4}{3}\)

\[
\int_0^1 \int_{[0,1]} z \, dA = \left( \int_0^1 x^2 \, dx \right) \left( \int_y^4 \, dy \right) \\
= \left( \frac{1}{3} \right) \left( \frac{4^2 - 1^2}{2} \right) \\
= \left( \frac{1}{3} \right) \left( \frac{15}{2} \right) \\
= \frac{\sqrt{5}}{2}
\]

9. Find the center of mass of a lamina (flat plate) occupying the square \(\mathcal{R} = [0,2] \times [-1,1]\) in the \(xy\) plane and having the density function \(\rho(x, y) = x\).

(A) \((0, 0)\)  
(B) \((0, \frac{2}{3})\)  
(C) \((0, \frac{3}{4})\)  
(D) \((0, -1)\)  
(E) \((\frac{2}{3}, 0)\)  
(F) \((1, 0)\)  
(G) \((\frac{4}{3}, 0)\)  
(H) \((\frac{3}{5}, 0)\)  
(I) \((\frac{6}{5}, 0)\)  
(J) \((1, \frac{2}{3})\)

By symmetry, \(\bar{y} = 0\) so \(\boxed{\bar{y} = 0}\)

\[
m = \int_0^2 x \int_{-1}^1 dy \, dx = \left( \frac{4}{2} \right) (2) = 4
\]

\[
\bar{x} = \frac{1}{4} \int_0^2 x^2 \int_{-1}^1 dy \, dx = \frac{\left( \frac{8}{3} \right) (2)}{4} = \boxed{\frac{4}{3}}
\]
10. If \( f(x, y) \) is any integrable function with values \( \geq 0 \), set up the integral expressing the volume under the surface \( z = f(x, y) \) and above the region in the \( xy \) plane bounded by \( y = x \) and \( x = y^2 - y \).

(A) \( \int_0^2 \int_x^y f(x, y) \, dy \, dx \)
(B) \( \int_0^2 \int_x^y f(x, y) \, dy \, dx \)
(C) \( \int_0^2 \int_{1/2 - \sqrt{x + 16}}^0 f(x, y) \, dy \, dx \)
(D) \( \int_0^2 \int_0^y f(x, y) \, dy \, dx \)
(E) \( \int_{-1/4}^{1/2} \int_{-y}^{y^2} f(x, y) \, dy \, dx \)
(F) \( \int_0^2 \int_{y^2 - y}^y f(x, y) \, dx \, dy \)
(G) \( \int_0^8 \int_{y^2 - y}^y f(x, y) \, dx \, dy \)
(H) \( \int_0^4 \int_{y^2 - y}^y f(x, y) \, dx \, dy \)
(I) \( \int_0^3 \int_{y^2 - y}^y f(x, y) \, dx \, dy \)
(J) \( \int_0^4 \int_{y^2 - y}^y f(x, y) \, dx \, dy \)

\[ \text{The boundary curves intersect when} \]
\[ y = x = y^2 - y \]
\[ 2y = y^2 \]
\[ \rightarrow y = 0 \]
\[ \rightarrow y = 2 = x \]

\[ V = \int_D f(x, y) \, dA = \int_0^2 \int_{x = y^2 - y}^{y^2} f(x, y) \, dx \, dy \]

11. Which of the following expresses the surface area of the surface parametrized by \( r(u, v) = \langle u \cos(v), u \sin(v), 4u \rangle \) for \( 0 \leq u \leq 2, 0 \leq v \leq 2\pi \).

(A) \( 4\pi \sqrt{13} \)
(B) \( \pi \sqrt{37} \)
(C) \( 2\pi \sqrt{5} \)
(D) \( 4\pi \sqrt{17} \)
(E) \( 16\pi \)
(F) \( 16\sqrt{17} \)
(G) \( 31\sqrt{5} \)
(H) \( 8\sqrt{13} \)
(I) \( 36 \)
(J) \( 49 \)

\[ \mathbf{r}_u = \langle \cos(v), \sin(v), 4u \rangle \quad \text{in the direction of} \]
\[ \mathbf{r}_v = \langle -u \sin(v), u \cos(v), 0 \rangle \quad \text{outwards} \]
\[ (\mathbf{r}_u \times \mathbf{r}_v) = (u^2 \mathbf{i} + u^2 \mathbf{j}) = u \mathbf{k} \]
\[ \mathbf{r}_u \times \mathbf{r}_v = 4u \sqrt{1 + u^2} \]

and \( A = \sqrt{17} \int_0^2 u \int_0^{2\pi} \mathbf{r}_u \times \mathbf{r}_v = 4\pi \sqrt{17} \)
12. Find the moment of inertia about the $x$-axis for a lamina (flat plate) occupying the region in the first quadrant bounded by the circle $x^2 + y^2 = 9$ and having density function $\rho(x, y) = x$.

(A) 10
(B) 21/2
(C) 78/7
(D) 44/3
(E) 72/5

\[
I_{x-axis} = \iint_{\Omega} y^2 \, dA
\]

In polar, \( \rho \) is described by \( 0 \leq \theta \leq \frac{\pi}{2} \)
\( 0 \leq r \leq 3 \)

\[
= \frac{3}{15} = \frac{1}{5}
\]

13. Suppose the joint probability density function $f(x, y, z)$ of a triple $(X,Y,Z)$ of random variables is 0 when $(x, y, z)$ is not in the cube $B = [0, 2] \times [0, 2] \times [0, 2]$ and is equal to a constant times $xyz$ when $(x, y, z)$ is in $B$. Find the probability that each of $X, Y, Z$ is $\leq 1$.

(A) $1/64$
(B) $1/16$
(C) $1/8$
(D) $1/4$
(E) $1/2$

\[
\begin{array}{l}
\text{(F) 0} \\
\text{(G) 5/12} \\
\text{(H) 3/8} \\
\text{(I) 15/16} \\
\text{(J) 1}
\end{array}
\]

Since \( \int_{x=0}^{2} x \, dx = 2 \), we must have

\[
f_X(x) = \begin{cases} 
\frac{1}{2} & 0 \leq x \leq 2 \\
0 & \text{otherwise}
\end{cases}
\]

and similarly for \( f_Y \) and \( f_Z \) with

\[
f(x, y, z) = f_X(x) f_Y(y) f_Z(z) = \begin{cases} 
\frac{1}{8} & 0 \leq x, y, z \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Then $P(X, Y, Z \text{ all } \leq 1) = \int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz$

\[
= \left( \frac{1}{8} \right)^3 = \left( \frac{1}{4} \right)^3 = \frac{1}{64}
\]
14. Find the area of the part of the plane \( z = 3x + 4y + 10 \) lying above the triangle in the \( xy \) plane with vertices (1,1), (1,5), and (3,3).

\[
\text{(A) } 6\sqrt{2} \\
\text{(B) } \sqrt{76} \\
\text{(C) } 2\sqrt{21} \\
\text{(D) } 3\sqrt{18} \\
\underline{\text{(E) } 4\sqrt{26}} \\
\text{(F) } 5\sqrt{5} \\
\text{(G) } 6\sqrt{18} \\
\text{(H) } 4\sqrt{30} \\
\text{(I) } 5\sqrt{29} \\
\text{(J) } 3\sqrt{47}
\]

15. Find the volume of the solid region bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 8 - (x^2 + y^2) \).

\[
\text{(A) } 6\pi \\
\text{(B) } 8\pi \\
\text{(C) } 10\pi \\
\text{(D) } 12\pi \\
\text{(E) } 14\pi \\
\underline{\text{(F) } 16\pi} \\
\text{(G) } 18\pi \\
\text{(H) } 20\pi \\
\text{(I) } 22\pi \\
\text{(J) } 24\pi
\]

The surfaces intersect when
\[
x^2 + y^2 = 8 - (x^2 + y^2) \\
x^2 + y^2 = 4
\]

In \( \rho \) space
\[
\frac{2\rho^2}{3} - 2\rho \cos \theta = 8 - 2\rho^2
\]

\[
\text{Vol} = \int_0^{2\pi} \int_0^2 (8 - 2\rho^2) \rho \, d\rho \, d\theta
\]

\[
= 2\pi \left\{ \int_0^2 8\rho \, d\rho - \int_0^2 2\rho^3 \, d\rho \right\}
\]

\[
= 2\pi \left\{ \left[ 4\rho^2 \right]_0^2 - \left[ \frac{2\rho^4}{4} \right]_0^2 \right\} = \left[ 16\pi \right]
\]
16. Which of the following integrals expresses the surface area of the part of the paraboloid \( z = x^2 + y^2 \) lying inside the cylinder \( x^2 + y^2 = 16 \).

(A) \( 2\pi \int_0^4 (r^6 \sqrt{1 + 4r}) \, dr \)

(B) \( \int_0^4 2x^3 \, dx \)

(C) \( \int_0^4 \int_0^{16-x^2} y \, dy \, dx \)

(D) \( 2\pi \int_0^4 (r \sqrt{1 + 4r^2}) \, dr \)

(E) \( \int_0^4 (x \sqrt{1 + 4x^2}) \, dx \)

(F) \( 2\pi \int_0^4 (\sqrt{1 + 2r^2}) \, dr \)

(G) \( 2\pi \int_0^{16} (r \sqrt{1 + 8r^2}) \, dr \)

(H) \( \int_0^{16} (\sqrt{1 + 4x^2}) \, dx \)

(I) \( 2\pi \int_0^2 (r \sqrt{4 + 4r^2}) \, dr \)

(J) \( 2\pi \int_0^4 (\sqrt{r^2 - 16}) \, dr \)

17. Suppose the two light bulbs in a lamp fixture burn out independently of one another and that the burn out times for each are exponentially distributed with a mean of 1000 hours. Find the probability that both of the burn out times will be \( \geq 2000 \) hours.

(A) \( 1/4 \)

(B) \( 1/8 \)

(C) \( 1/16 \)

(D) \( 1/2 \)

(E) \( (1 - e^{-2})^2 \)

(F) \( e^2 - 1 \)

(G) \( (1 - e^{-1})^2 \)

(H) \( e^{-2} \)

(I) \( e^{-4} \)

(J) \( e^{-8} \)

\[ P(both \ \Sigma \ and \ \Xi \geq 2000) = P(\xi \geq 2000) \cdot P(\xi \geq 2000) \]

\[ = \left( \int_{2000}^{\infty} e^{-\frac{x}{1000}} \, dx \right)^2 \]

\[ = \left( \int_{2000}^{\infty} e^{-u} \, du \right)^2 \]

\[ = \left( e^{-2} \right)^2 \]

\[ = e^{-4} \]
18. Find the equation of the tangent plane at \((4, -1, 1)\) for the ellipsoid
\[ x^2 + 2y^2 + 3z^2 = 21. \]

\[ \nabla F(4, -1, 1) = \langle 8, -4, 6 \rangle \]
\[ = 2 \langle 4, -2, 3 \rangle \]
So \(\langle 4, -2, 3 \rangle\) is a normal vector for the tangent plane and the equation of the plane is
\[ 0 = \langle 4, -2, 3 \rangle \cdot \langle x-4, y+1, z-1 \rangle \]
\[ = 4(x-4) - 2(y+1) + 3(z-1) \]

19. Find the volume above the \(xy\) plane and below the paraboloid
\[ z = 4 - (x^2 + y^2) \]

\[ (A) 4\pi \]
\[ (B) 8\pi \]
\[ (C) 12\pi \]
\[ (D) 16\pi \]
\[ (E) 20\pi \]
\[ \text{(F) } 24\pi \]
\[ \text{(G) } 28\pi \]
\[ \text{(H) } 32\pi \]
\[ \text{(I) } 36\pi \]
\[ \text{(J) } 40\pi \]
\[ \text{vol} = \iint_{x^2+y^2\leq 4} z \, dA = \iiint_{0}^{2} (4-r^2) \, r \, dr \, d\theta \]
\[ = 2\pi \left\{ \int_{0}^{2} r^2 \, dr - \int_{0}^{2} r \, dr \right\} \]
\[ = 2\pi \left\{ \frac{8}{3} - \frac{16}{3} \right\} \]
\[ = 8\pi \]
20. Which of the following sets of inequalities describes the tetrahedron with vertices at \((0,0,0), (0,1,0), (1,1,0),\) and \((0,1,1)\)? Note that the mutually perpendicular planes \(z = 0, \ x = 0,\) and \(y = 1\) meet at \((0,1,0)\) and form 3 of the 4 boundaries for the tetrahedron.

(A) \(0 \leq y \leq 1, \ y \leq x \leq 1, \ 0 \leq z \leq x - y\)
(B) \(0 \leq y \leq 1, \ 0 \leq x \leq y, \ 0 \leq z \leq y - x\)
(C) \(0 \leq y \leq 1, \ 0 \leq x \leq y, \ 0 \leq z \leq x + y\)
(D) \(0 \leq x \leq 1, \ 0 \leq y \leq x, \ 0 \leq z \leq x - y\)
(E) \(0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq x + y\)
(F) \(0 \leq y \leq 1, \ 0 \leq x \leq 1, \ 0 \leq z \leq y - x\)
(G) \(0 \leq x \leq 1, x \leq y \leq 1, \ 0 \leq z \leq x + y\)
(H) \(0 \leq x \leq 1, x \leq y \leq 1 - x, \ 0 \leq z \leq 1\)
(I) \(0 \leq y \leq 1, y \leq x \leq 1 - y, \ 0 \leq z \leq y - x\)
(J) \(0 \leq z \leq 1, \ 0 \leq y \leq 1, \ 0 \leq x \leq z + y\)

The equation of the plane through \((0,0,0), (1,1,0)\) and \((0,1,1)\) is \(z = y - x\). The tetrahedron lies below this plane and above the triangle with vertices \((0,0), (0,1), (1,1)\). The description of the plane is

\[0 \leq y \leq 1\]
\[0 \leq x \leq y\]
\[0 \leq z \leq y - x\]
Problems 21 and 22 will be hand graded. Write your name and ID# in the slots above and on the following page. Show all your work in the space below the problem, using the reverse side if more space is needed. Answers alone with no supporting work will not receive credit.

21. Consider the function \( f(x, y) = x^2 + y^2 - 2x + 2y + 7 \) on the disk \( D \) where \( x^2 + y^2 \leq 4 \).

(i) [6 points] Find the critical point of \( f \) in the interior of \( D \). What type of critical point (local maximum, local minimum, saddle point, or none of these) is it?

(ii) [6 points] Set up and solve equations determining boundary points where \( f \) assumes extreme values.

(iii) [3 points] Use your answers to (i) and (ii) to find the maximum and minimum values of \( f \) on the domain \( D \).

N.B. There are many valid ways to approach this problem including substitutions of various kinds. Some methods will involve Lagrange multiplier equations in (ii), others will not. You are free to use any method you wish provided your approach is clearly indicated.

\[
\begin{align*}
(1) & \quad \nabla f = \nabla (x^2 + y^2 - 2x + 2y + 7) \\
& \quad (1,1) \text{ is the critical point in the interior of } D \\
& \quad 0 = f_x = 2x - 2 \\
& \quad 0 = f_y = 2y + 2 \\
& \quad f_{xx}(1,1) = 2 = f_{yy}(1,1) \\
& \quad \text{and } f_{xy}(1,1) = 0 \quad \text{so } D = f_{xx}f_{yy} - f_{xy}^2 \geq 0 \\
& \quad \text{and } f \text{ has a local minimum at } (1,1). \\
& \quad \text{Actually, } f \text{ has a global minimum at } (1,1). \\
\end{align*}
\]

Since \( f(x, y) = (x-1)^2 + (y+1)^2 + 5 \geq 5 \) with equality only if \( x = 1, y = -1 \)

\[
(\text{ii}) \quad \nabla f = \lambda \nabla (x^2 + y^2) \\
2(x-1, y+1) = 2\lambda (x, y) \\
\text{and } x^2 + y^2 = 4
\]

We can't have \( x = y = 0 \) since \( x = 0 \Rightarrow x-1 = 0 \Rightarrow \lambda = 0 \)

Hence \( x-1 = \lambda = y+1 \Rightarrow \frac{-1}{x} = y \Rightarrow x = -1 \)

Then \( x^2 + x^2 + y^2 = 4 \Rightarrow x^2 = 2 \) and

either \( x = \sqrt{2}, y = -\sqrt{2} \) or \( x = -\sqrt{2}, y = \sqrt{2} \)

\[
(\text{iii}) \quad f(\sqrt{2}, -\sqrt{2}) = 4 + 7 - 4\sqrt{2} \quad \text{and} \quad f(-\sqrt{2}, \sqrt{2}) = 11 + 4\sqrt{2}
\]

Since \( 11 + 4\sqrt{2} > 11 - 4\sqrt{2} > 5 \), \( 11 + 4\sqrt{2} \) is the maximum value and 5 is the minimum value.
22. As in the problem considered in class, suppose Xavier and Yolanda agree to meet shortly after noon for coffee and that their arrival times $X$ and $Y$ are independent of each other with $X$ being exponentially distributed with mean 1 and $Y$ having the probability density function $f_Y(y)$ which is equal to $y/50$ for $0 \leq y \leq 10$ and is otherwise zero.

(i) [4 points] What is Yolanda's mean arrival time?

(ii) [5 points] What is the probability that Xavier arrives within the first minute after noon and Yolanda arrives between 5 and 10 minutes after noon?

(iii) [6 points] What is the probability $P(X \leq Y)$ that Xavier arrives before Yolanda?

\[
(i) \quad f_Y(y) = \begin{cases} \frac{y}{50} & 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}
\]

\[
\bar{Y} = \text{Yolanda's mean time} = \frac{1}{\frac{50}{2}} \int_0^{10} y \, dy
\]

\[
= \frac{1}{\frac{50}{2}} \left( \frac{5}{2} \right) = \frac{10}{3} = \frac{\sqrt{16}}{3}
\]

\[
(i i) \quad P(X \leq 1, 5 \leq Y \leq 10) = P(X \leq 1) \cdot P(5 \leq Y \leq 10)
\]

\[
= (1 - e^{-1}) \cdot \frac{1}{50} \int_5^{10} y \, dy
\]

\[
= \sqrt{\frac{2}{3}} (1 - e^{-1})
\]

\[
(i i i) \quad P(X \leq Y)
\]

\[
= \frac{1}{50} \int_0^{10} y \int_0^y e^{-x} \, dx \, dy
\]

\[
= \frac{1}{50} \left[ -y e^{-y} + y + 1 \right]_0^{10} \, dy
\]

\[
= 1 - \frac{1}{50} \int_0^{10} ye^{-y} \, dy
\]

\[
= 1 - \left[ -ye^{-y} - e^{-y} \right]_{y=0}^{y=10}
\]

\[
= \frac{49}{50} + 11e^{-10} \approx 0.98
\]