Solutions

FINAL EXAM, MATH 233
FRIDAY, DECEMBER 19, 2003

This examination has 25 multiple choice questions, each worth 4 points for a maximum of 100 points. Please check over your exam booklet and, if you find it to be incomplete, notify the proctor. Do all your supporting calculations in this booklet. In case of a doubtful mark on your answer card, your instructor can then check here. When you mark your card, use a soft lead pencil (#2). Erase fully any answers you want to change.

You may use one 3 by 5 note card and a scientific calculator.

(1) The center and radius of the sphere \( x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0 \) are

(A) \((4, 1, 3)\) and 2
(B) \((2, 1, -3)\) and 5
(C) \((4, -1, -3)\) and 5
(D) \((4, -13, -3)\) and 5
(E) \((14, -1, -3)\) and 5
(F) \((2, -1, -1)\) and 2
(G) \((14, -1, -13)\) and 3
(H) \((24, -2, -3)\) and 5
(I) \((4, -1, -4)\) and 5
(J) \((4, -0, -3)\) and 15

\[
\begin{align*}
(x - 4)^2 + (y + 1)^2 + (z + 3)^2 &= -1 + 4^2 + 1^2 + 3^2 \\
&= 16 + 9 \\
&= 25 \\
&= 5^2
\end{align*}
\]

Center: \((4, -1, -3)\)
Radius: 5
(2) Parametric equations for the line passing through \((1, 2, 4)\) and in the direction of the vector \(<2, -1, 3>\) are:

(A) \(x = 1 + t, y = t, z = 4 - 3t\)
(B) \(x = 12 + t, y = 2 + t, z = 4 + 3t\)
(C) \(x = 1 + t, y = 2 + t, z = -3t\)
(D) \(x = 2 + 2t, y = 2 - 2t, z = 1 - 3t\)
(E) \(x = 11 - 2t, y = 2 + t, z = 4 - 3t\)
(F) \(x = 3 + 2t, y = 23 - t, z = 4 + t\)
(G) \(x = 1 + 2t, y = 2 - t, z = 4 + 3t\)
(H) \(x = 1 + 4t, y = 2 - t, z = 4 + 3t\)
(I) \(x = 1 + 2t, y = -t, z = 4 + 3t\)
(J) \(x = 1 + 2t, y = 2 - t, z = 3t\)

\[
\langle x, y, z \rangle = \langle 1, 2, 4 \rangle + t \langle 2, -1, 3 \rangle
\]

\[
\begin{align*}
x &= 1 + 2t \\
y &= 2 - t \\
z &= 4 + 3t
\end{align*}
\]

(3) The equation of the plane through the point \((-4, 1, 2)\) and parallel to the plane \(x + 2y + 3z = 3\) is

(A) \(x + 2y + 3z = 4\)
(B) \(5x - 2y + z = 18\)
(C) \(3x + 4y + 3z = 1\)
(D) \(x - 2y + 3z = 7\)
(E) \(x + y + z = 2\)
(F) \(x - 2y + z = 8\)
(G) \(x + 21y + 5z = 18\)
(H) \(3x + y + 13z = 3\)
(I) \(x + 2y - z = 2\)
(J) \(x - 2y + 3z = 1\)

**Parallel planes have the same normal vectors**, so \(\vec{n} = \langle 1, 2, 3 \rangle\) is a normal vector for the desired plane.

**Equation:** \(x + 2y + 3z = \vec{n} \cdot \langle x, y, z \rangle = \vec{n} \cdot (-4, 1, 2) = -4 + 2 + 6 = 4\)
(4) The line \( x = 2 - t, y = 1 + 3t, z = 4t \) intersects the plane \( 2x - y + z = 2 \) at

(A) \((4, -1, -4)\)
(B) \((2, 1, -3)\)
(C) \((4, -1, -3)\)
(D) \((4, -13, -3)\)
(E) \((14, -1, -3)\)
(F) \((2, -1, -1)\)
(G) \((14, -1, -13)\)
(H) \((24, -2, -3)\)
(I) \((1, 4, 4)\)
(J) \((4, -0, -3)\)

\[
2 \left( \frac{2-t}{x} \right) - \left( \frac{1+3t}{y} \right) + \left( \frac{4t}{z} \right) = 2
\]

\[
3 - 2t - 3t + 4t = 3 - t = 2
\]

\[
t = 1
\]

Intersection Point:

\[
x = 2 - 1 = 1
\]
\[
y = 1 + 3 = 4
\]
\[
z = 4 \cdot 1 = 4
\]

(5) Find the tangential and normal components of the acceleration vector at \( t = 0 \) for a body whose position vector at any time \( t \) is given by \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \).

(A) \(2, 3\)
(B) \(0, 1\)
(C) \(1, 5\)
(D) \(1, 0\)
(E) \(0, 0\)
(F) \(-1, 0\)
(G) \(-1, -1\)
(H) \(2, 3\)
(I) \(-3, 4\)
(J) none of the above

\[
\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle
\]

\[
\mathbf{r}^\prime(0) = \langle 0, 1, 1 \rangle
\]

\[
\mathbf{r}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle
\]

\[
\mathbf{a} = \mathbf{r}''(0) = \langle -1, 0, 0 \rangle
\]

\[
\mathbf{a}_t = \mathbf{a} \cdot \mathbf{r}^\prime(0) = \mathbf{a} \cdot \langle 0, 1, 1 \rangle = 0
\]

Therefore \( \mathbf{a} = \mathbf{a}_t + \mathbf{a}_n \mathbf{r}^\prime(0) = \mathbf{a}_n \mathbf{r}^\prime(0) \)

and \( \mathbf{a}_n = \frac{1}{|\mathbf{a}|} = 1 \)
(6) The length $l$, width $w$, and height $h$ of a box are changing with time. Assume that distances are measured in meters and time in seconds and at a certain time $t_0, w = 1, l = h = 2$ and $dl/dt = .5, dw/dt = 1, dh/dt = -.5$ Find the rate in meters per second at which the diagonal length $s = \sqrt{l^2 + w^2 + h^2}$ is changing at time $t_0$.

(A) $2/9$
(B) $1/3$
(C) $1/2$
(D) $7/9$
(E) $1$
(F) $4/3$
(G) $5/9$
(H) $3/4$
(I) $2$
(J) $7/3$

\[
\frac{ds}{dt} = \frac{l \frac{dl}{dt} + w \frac{dw}{dt} + h \frac{dh}{dt}}{\sqrt{l^2 + w^2 + h^2}}
\]

\[
\frac{ds}{dt}(t_0) = \frac{2(1) + 1(2) + 2(-.5)}{3} = \frac{1}{3} \text{ m/sec}
\]

(7) Let $f(x, y, z) = \sin(3x + yz)$. Then $f_{yz}$ equals:

(A) $f_{yz} = -yz \sin(3x + yz)$
(B) $f_{yz} = -\sin(3x + yz)$
(C) $f_{yz} = -y \sin(3x + yz)$
(D) $f_{yz} = -z \sin(3x + yz)$
(E) $f_{yz} = yz \sin(3x + yz)$

\[
\frac{\partial f}{\partial y} = f_y = z \cos(3x + yz)
\]

\[
\frac{\partial f}{\partial z} = f_z = 3 \cos(3x + yz)
\]

\[
f_{yz} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} = \cos(3x + yz) - yz \sin(3x + yz)
\]
(8) A surface has parametric equations: \( x = u + v, \ y = 3u^2, \ z = u - v \). Its tangent plane at the point \( (2, 3, 0) \) has the equation:

(A) \( 5(x - 2) + 2(y - 3) - 6z = 0 \);
(B) \( -6(x - 2) - (y - 3) + 12z = 0 \);
(C) \( 3(x - 2) + (y - 3) - 2z = 0 \);
(D) \( (x - 2) - 2(y - 3) + 4z = 0 \);
(E) \( -3(x - 2) + (y - 3) - 3z = 0 \);
(F) \( -6(x - 2) + 2(y - 3) - 3z = 0 \);
(G) \( 12(x - 2) + (y - 3) - 5z = 0 \);
(H) \( -6(x - 2) - 4(y - 3) + z = 0 \);
(I) \( (x - 2) - 7(y - 3) + 8z = 0 \);
(J) \( 4(x - 2) + (y - 3) - 9z = 0 \).

\[
\begin{align*}
\{ x = 2 = u + v, \\ y = 3 = 3u^2, \\ z = 0 = u - v \} \\
to \quad \vec{n}_u(1,1,1) &= \langle 1, 6, 1, 1 \rangle \\
to \quad \vec{n}_v(1,1,1) &= \langle 1, 0, -1 \rangle \\
\vec{n}_u(1,1,1) \times \vec{n}_v(1,1,1) &= \langle -6, 2, -6 \rangle \\
 &= 2 \langle -3, 1, -3 \rangle \\
\text{is a normal vector for the tangent plane at } (2, 3, 0). \\
\text{Equation for the tangent plane:} \\
-3(x - 2) + (y - 3) - 3z &= 0
\end{align*}
\]

(9) Find the maximum value of the function \( f(x) = x + y - z \) for points satisfying the constraint \( x^2 + y^2 + z^2 = 6 \).

(A) 0
(B) \( \sqrt{2} \)
(C) \( 2\sqrt{3} \)
(D) \( 3\sqrt{2} \)
(E) 3
(F) \( 4\sqrt{3} \)
(G) 6
(H) \( 6\sqrt{2} \)
(I) \( 6\sqrt{3} \)
(J) 12

\[\nabla f = \langle 1, 1, -1 \rangle = 2\lambda \langle x, y, z \rangle = \lambda \nabla(x^2 + y^2 + z^2)\]

\[x^2 + y^2 + z^2 = 6\]

\[x = y = -z = (\frac{1}{2\lambda})\]

\[x^2 = y^2 = z^2 = 2\]

The maximum value of \( f \) is \( 3\sqrt{2} \) and is obtained when \( x = y = -z = \sqrt{2} \).
(10) Let \( D \) be the plane region in the first quadrant that lies between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \). Find

\[
\iint_D xy \, dA.
\]

(A) \( \frac{3}{8} \)
(B) \( \frac{7}{8} \)
(C) \( \frac{15}{8} \)
(D) \( 2 \)
(E) \( \frac{25}{16} \)
(F) \( \frac{11}{4} \)
(G) \( 3 \)
(H) \( \frac{3\pi}{16} \)
(I) \( \frac{7\pi}{32} \)
(J) \( \frac{15\pi}{64} \)

In polar coordinates, \( x = r \cos \theta \), \( y = r \sin \theta \), \( D \) is described by \( 1 \leq r \leq 2 \), \( 0 \leq \theta \leq \frac{\pi}{2} \) and \( dA = r \, dr \, d\theta \)

\[
\begin{align*}
\iint_D xy \, dA &= \int_0^{\pi/2} \int_1^2 (r \cos \theta)(r \sin \theta) \, r \, dr \, d\theta \\
&= \int_0^{\pi/2} \int_1^2 r^3 \cos \theta \sin \theta \, dr \\
&= \left[ -\frac{r^2}{2} \right]_1^2 \cos \theta \sin \theta \\
&= \frac{3}{2} \cos \theta \sin \theta \\
&= \left( \frac{1}{2} \right) \left( \frac{1}{4} \right) \\
&= \frac{15}{8}
\end{align*}
\]

(11) Find the directional derivative of the function \( f(x, y) = x^3 y + 2xy^2 \) at the point \((1,1)\) in the direction of the vector \( \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle \)

(A) 0
(B) 3
(C) \( 2\sqrt{3} \)
(D) \( 5\sqrt{5} \)
(E) 4
(F) \( 4\sqrt{2} \)
(G) \( 5\sqrt{2} \)
(H) \( 3\sqrt{3} \)
(I) 10
(J) 12

\[
\nabla f = \langle 3x^2 y + 2y^2, x^3 + 4xy \rangle
\]

\[
\nabla f (1,1) = \langle 5, 5 \rangle
\]

\[
\nabla f (1,1) \cdot \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle = \frac{10}{\sqrt{2}} = 5\sqrt{2}
\]
(12) Find \( \iiint_E (x^2 + y^2 + z^2) \, dV \), where \( E \) is the region in the first octant lying below the sphere \( x^2 + y^2 + z^2 = 4 \).

(A) \( \frac{8\pi}{3} \)
(B) \( \frac{15\pi}{6} \)
(C) \( \frac{13\pi}{3} \)
(D) \( \frac{29\pi}{15} \)
(E) \( \frac{15\pi}{4} \)
(F) \( \frac{27\pi}{10} \)
(G) \( \frac{16\pi}{9} \)
(H) \( \frac{25\pi}{8} \)
(I) \( \frac{24\pi}{7} \)
(J) \( \frac{16\pi}{5} \)

In spherical coordinates,
\[
(x^2 + y^2 + z^2) \, dV = (\rho^2)(\rho^2 \sin \phi \, d\rho \, d\phi \, d\Theta)
\]
and \( E \) is described by
\[
0 \leq \rho \leq 2 \quad \quad 0 \leq \phi \leq \frac{\pi}{2} \quad \quad 0 \leq \Theta \leq \frac{12\pi}{5}
\]

\[
\iiint_E (x^2 + y^2 + z^2) \, dV = \left( \int_0^{\pi/2} d\Theta \right) \left( \int_0^{\pi/2} \sin \phi \, d\phi \right) \int_0^2 \rho^4 \, d\rho
\]
\[
= \left( \frac{\pi}{2} \right) \left( 1 \right) \left( \frac{2^5}{5} \right)
\]
\[
= \frac{32\pi}{10} = \frac{16\pi}{5}
\]

(13) Find the area enclosed by the ellipse \( \frac{x^2}{9} + \frac{y^2}{25} = 1 \).

(A) \( 3\pi \)
(B) \( 5\pi \)
(C) \( 25\pi \)
(D) \( 9\pi \)
(E) \( 10\pi \)
(F) \( 12\pi \)
(G) \( 13\pi \)
(H) \( 14\pi \)
(I) \( 15\pi \)
(J) \( 16\pi \)

Using the substitution \( u = \frac{x}{3}, \ v = \frac{y}{5} \)

transforms \( R : (\frac{x^2}{9} + (\frac{y^2}{25}) \leq 1 \)

\( R : u^2 + v^2 \leq 1 \)

with \( \Omega(x, y) = \det \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = 15 \)

Area of \( R \) = \( \iint_{R^*} dA = \iint_2 15 \, d\tilde{A} \)

\[
= 15 \left( \text{area of disk of radius 1} \right) = 15\pi
\]
(14) A force field $\mathbf{F}$ is given by $\mathbf{F}(x, y) = \langle y, x \rangle$. Under the influence of $\mathbf{F}$, an object moves along the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. Find the work done on the object by $\mathbf{F}$.

\[ \text{Work done} = \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C} y \, dx + x \, dy \]

\[ = \int_{-3}^{2} \left\{ y \left( \frac{-2y}{\frac{d}{x} \cdot \frac{dy}{x}} \right) + \left( \frac{4 - y^2}{x} \right)^2 \right\} \, dy \]

\[ = 4 \int_{-3}^{5} \, dy - 3 \int_{-3}^{5} \, dy \]

\[ = 4 \cdot 5^2 - (2^3 - (-3)^3) \]

\[ = 20 - 8 - 27 \]

\[ = -15 \]

(15) Given that $\mathbf{F}(x, y) = \langle 2x, 2y \rangle$, compute the line integral $\int_{C} \mathbf{F} \cdot d\mathbf{r}$, where $C$ is the parametric curve defined as follows: $x = t \cos(t), y = t \sin(t), 0 \leq t \leq 2\pi$.

\[ \mathbf{F} = \langle 2x, 2y \rangle = \nabla(x^2 + y^2) \text{ is conservative} \]

\[ C \text{ has endpoints } A(0, 0) \text{ and } B(2\pi, 0) \]

\[ \left. \left( x^2 + y^2 \right) \right|_{(0, 0)}^{(2\pi, 0)} \]

\[ = (2\pi)^2 \]

\[ = 4\pi^2 \]
(16) \( \mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle \) is a conservative vector field in 3-dimensional space. Which of the following is a potential function of \( \mathbf{F} \)?

(A) \( f(x, y, z) = x + y + z \)
(B) \( f(x, y, z) = xy + yz + xz \)
(C) \( f(x, y, z) = 0 \)
(D) \( f(x, y, z) = x^2 + y^2 + z^2 \)
(E) \( f(x, y, z) = xyz(x + y + z) \)
(F) \( f(x, y, z) = 2(x + y + z) \)
(G) \( f(x, y, z) = x^3 + y^3 + z^3 \)
(H) \( f(x, y, z) = xyz \)
(I) \( f(x, y, z) = x^2y^2z^2 \)
(J) \( f(x, y, z) = x^3y^3z^3 \)

\( \nabla \cdot (\vec{\omega}) = yz, xz, xy \rangle = \nabla (xyz + C) \)

for any constant \( C \)

(17) Let \( \mathbf{F}(x, y, z) = \langle \sin^3 x e^y, \cos \sqrt{x^2 + y^3 + z/2}, 3x \rangle \), then \( \text{div}(\text{curl}(\mathbf{F})) \) at the point \((1, 1, 1)\) is

(A) 2
(B) \( 5e \)
(C) \( 1/e \)
(D) 0
(E) \( e \)
(F) 2
(G) 3
(H) \( e^2 - \sqrt{e} \)
(I) \( \sqrt{3e} \)
(J) \( \sqrt{3e} \)

\( \text{div}(\text{curl}(\vec{\omega})) = 0 \) for every twice continuously differentiable \( \vec{\omega} \).
(18) Find a vector field \( \mathbf{G} \) such that \( \text{curl} \, \mathbf{G} = < 2x, 3yz, -xz^2 > \).

(A) \(< x^3, xz, y^2 > \)
(B) \(< x, y - yz > \)
(C) \(< yz, xy, xz > \)
(D) \(< xz, -1, xyz > \)
(E) \(< z^3y, z^2x, z > \)
(F) \(< xy^2z, 3y, xz > \)
(G) \(< 3y, x^2, x^3 > \)
(H) \(< -x, -y, z > \)
(I) \(< y^2z, xz, x^3 > \)
(J) \( \mathbf{G} \) does not exist.

Let \( \mathbf{E} = < 2x, 3yz, -xz^2 > \)

Then \( \text{div} \, \mathbf{E} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3yz) + \frac{\partial}{\partial z}(-xz^2) \)

\[ = 2 + 3z - 2xz \]

is not always 0.

Since \( \text{div} \, \text{curl} \, \mathbf{E} = 0 \) for all twice cont. differentiable \( \mathbf{E} \),

there is no \( \mathbf{G} \) for which \( \mathbf{E} = \text{curl} \, \mathbf{G} \).

(19) Evaluate \( \int_C e^y \, dx + 2xe^y \, dy \) where \( C \) is the curve, with counter-clockwise orientation, which bounds the square with sides \( x = 0, x = 1, y = 0, \) and \( y = 1. \)

(A) 2
(B) \( e + 5 \)
(C) \( 3 - 5e \)
(D) \( 1 + e^2 \)
(E) \( e^2 - 1 \)
(F) \( 3e \)
(G) 2
(H) 25
(I) \( e - 1 \)
(J) \( e + 2 \)

\[ \int_C e^y \, dx + 2xe^y \, dy = \iint_D \left( \frac{\partial}{\partial x}(2xe^y) - \frac{\partial}{\partial y}(xe^y) \right) \, dA \]

\[ = \left( \int_{x=0}^{1} \int_{y=0}^{1} e^y \, dy \right) \, dx \]

\[ = \left( \int_{x=0}^{1} \right) \, dx \cdot (e - 1) \]

\[ = e - 1 \]
(20) If \( \mathbf{F} = \langle e^{x^2}, \sin y \rangle \), then for any closed curve \( C \), \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is

(A) \( e - 1 \)
(B) \( \frac{2}{3e} + 1 \)
(C) \( 3 \)
(D) \( \pi \)
(E) \( \frac{9}{3} \)
(F) \( 0 \)
(G) \( e \)
(H) \( 24 \)
(I) \( 2 \)
(J) \( -1 \)

\[ \frac{\partial}{\partial x} (\sin y) - \frac{\partial}{\partial y} (e^{x^2}) = 0 \]

\( \mathbb{F} = \nabla (\varphi(x) - \cos y) \) is conservative with \( \varphi(x) \) an anti-derivative of \( e^{x^2} \).

Therefore, \( \int_C \mathbb{F} \cdot d\mathbf{r} = 0 \) for every closed curve \( C \).

(21) Find \( \int_C x \, dy \) where \( C \) is the counter-clockwise oriented curve bounding the rectangle with vertices \((1, -2), (3, -2), (3, 1), \) and \((1, 1)\)

(A) \( 0 \)
(B) \( 1 \)
(C) \( 2 \)
(D) \( 3 \)
(E) \( 4 \)
(F) \( 5 \)
(G) \( 6 \)
(H) \( 7 \)
(I) \( 8 \)
(J) \( 9 \)

\[ \mathcal{C} = \partial \mathcal{D} \]

\( C \) is the boundary \( \partial \mathcal{D} \) of the rectangle \( \mathcal{D} = [1, 3] \times [-2, 1] \)

\[ \int_C x \, dy = \iint_D \frac{\partial}{\partial x} (x) \, d\mathcal{A} = \iint_D d\mathcal{A} \]

\[ = \text{area of } \mathcal{D} \]
\[ = 3.2 = 6 \]
(22) Find \( \int_C xy\,dx + 2x^2\,dy \) where \( C \) is the counter-clockwise oriented curve consisting of the line segment from \((0,0)\) to \((2,0)\) followed by the quarter circle \( x^2 + y^2 = 4 \) from 
(2,0) to \((0,2)\) followed by the line segment from \((0,2)\) to \((0,0)\).

(A) 0  
(B) 1  
(C) 2  
(D) 3  
(E) 4  
(F) 5  
(G) 6  
(H) 7  
(I) 8  
(J) 9

\[ C = \partial D \]  
the boundary of the 
quarter disk \( D : \)  
\[ x^2 + y^2 \leq 4 \]  
\[ 0 \leq x, y \]

\[ \int_C xy\,dx + 2x^2\,dy = \iint_D \left\{ \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy) \right\} \, dA \]

\[ = 3 \int_D x \, dA \]

(\text{pass the polar})

\[ = 3 \int_0^{\pi/2} \frac{\rho}{2} \int_0^2 \rho \cos \theta \, \rho \, d\theta \, d\rho \]

\[ = 3 \left( \int_0^{\pi/2} \cos \theta \, d\theta \right) \left( \int_0^2 \rho^2 \, d\rho \right) \]

\[ = 1 \cdot 2^3 = 8 \]

(23) Find \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(x,y) = x^2\mathbf{i} - xy\mathbf{j} \) and \( C \) has the vector parametrization \( \mathbf{r}(t) = t^2\mathbf{i} + tj, 0 \leq t \leq 1 \).

(A) 0  
(B) 1/28  
(C) 1/14  
(D) 3/28  
(E) 1/7  
(F) 5/28  
(G) 3/14  
(H) 1/4  
(I) 4/14  
(J) 9/28

Using \( x(t) = t^2 \), \( \frac{dx}{dt} = 2t \)

\( y(t) = t \), \( \frac{dy}{dt} = 1 \)

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \{ (t^2)^2 \cdot t - t^2 \cdot t \} \, dt \]

\[ = \int_0^1 \{ t^5 - t^3 \} \, dt \]

\[ = \frac{t^6}{6} - \frac{t^4}{4} \bigg|_0^1 \]

\[ = \frac{1}{6} - \frac{1}{4} \]

\[ = \frac{8 - 7}{28} = \frac{1}{28} \]
(24) Let $\mathbf{F}(x, y, z) = \langle x^2y, -x^2y^2, -x^2yz \rangle$. Which of the following statements about $\mathbf{F}$ are true?

I. $\text{curl } \mathbf{F} = \langle 0, 0, 0 \rangle$

II. $\text{div } \mathbf{F} = 0$

III. $\mathbf{F}$ is a conservative vector field.

IV. Line integrals involving $\mathbf{F}$ are path independent.

(A) I and III

(B) I and II

(C) I, III, and IV

(D) II and III

(E) only I

(F) only II

(G) only III

(H) only IV

(I) all four

(J) none

\[
\begin{align*}
\text{rot } \mathbf{F} &= \frac{\partial}{\partial x} (x^2y) - \frac{\partial}{\partial y} (x^2y^2) - \frac{\partial}{\partial z} (x^2yz) \\
&= 3x^2y - 2x^2y - x^2y = 0 \\
\text{However, curl } \mathbf{F} &\neq \langle 0, 0, 0 \rangle \\
\text{since } \frac{\partial}{\partial x} (-x^2y^2) &\neq \frac{\partial}{\partial y} (x^2y), \text{ etc.} \\
\text{Therefore } \mathbf{F} \text{ is not conservative and line integrals of } \mathbf{F} \text{ are not path independent.}
\end{align*}
\]

(25) Evaluate $\int_C (x^2 + y^2) \, ds$ where $C$ is the curve parametrized by

$\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle, \quad 0 \leq t \leq \pi/2$

(A) $4\pi$

(B) $5\pi$

(C) $9\pi/2$

(D) $25\pi/2$

(E) $15\pi$

(F) $25\pi$

(G) $50\pi/3$

(H) $45\pi/2$

(I) $36\pi$

(J) $50\pi$

\[
\begin{align*}
x^2 + y^2 &= (3\cos t)^2 + (3\sin t)^2 = 9 \\
\frac{d\mathbf{r}}{dt} &= \langle -3\sin t, 3\cos t, 4 \rangle \\
\frac{d\mathbf{r}}{dt} &= \langle -3\sin t, 3\cos t, 4 \rangle \\
\frac{ds}{dt} &= \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = \sqrt{9 + 9 + 16} = 5 \\
\int_C (x^2 + y^2) \, ds &= \int_0^{\pi/2} (9) \, 5 \, dt \\
&= 45 \int_0^{\pi/2} \, dt \\
&= 45 \frac{\pi}{2}
\end{align*}
\]