Hankel vector moment sequences

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Pick functions

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A classical *Pick function* is an analytic function h on Π with nonnegative imaginary part, that is

$$\operatorname{Im} h(z) \geq 0, \text{ for } z \in \Pi.$$

Let $\mathcal{P}$ be the class of Pick functions.

Nevanlinna representations

Theorem (Nevanlinna)

Let h be a function defined on Π. There exists a finite positive measure μ on $\mathbb{R}$ such that

$$h(z) = \int \frac{\mathrm{d}\mu(t)}{t - z}$$

if and only if

1. $h \in \mathcal{P}$, *and*
2. $$\liminf_{y \to \infty} y \, |h(iy)| < \infty.$$

Connection to moment problems

Note that

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So, if μ is compactly supported,

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Thus, the moments of μ, $\int t^n d\mu(t)$ are given by the residues at ∞ of h.

Connection to moment problems II

Theorem (Nevanlinna)

Let ρ_n be a sequence of real numbers. There is a measure μ with moments

$$\rho_n = \int t^n d\mu(t)$$

if and only if there is a Pick function so that for each N

$$h(z) = -\sum_{n=0}^{N} \frac{\rho_n}{z^{n+1}} + o\left(\frac{1}{|z|^{n+1}}\right)$$

nontangentially at infinity.

Operator theoretic approach

An operator theoretic approach can be guided by the observation that the integral representation

$$h(z) = \int \frac{\mathrm{d}\mu(t)}{t - z}$$

can be written

$$h(z) = \left\langle (A - z)^{-1}\mathbf{1}, \mathbf{1} \right\rangle$$

where A is multiplication by t on $L^2(\mu)$ and $\mathbf{1}$ is the constant function 1.

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Note that

$$\langle A^n \mathbf{1}, \mathbf{1} \rangle = \int t^n d\mu.$$

Pick functions in two variables

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The 2-variable Pick class $\mathcal{P}_2$ is the set of analytic functions h on Π^2 such that

$$\operatorname{Im} h \geq 0.$$

2-variable type I Nevanlinna representation

Theorem (Agler, Tully-Doyle, Young)

For a function h defined on Π^2, there exists a Hilbert space $\mathcal{H}$, a densely defined self-adjoint operator A on $\mathcal{H}$, a vector $v \in \mathcal{H}$ and a positive contraction Y on $\mathcal{H}$ such that h has a type I representation

$$h(z) = \left\langle (A - z_1 Y - z_2(1 - Y))^{-1} v, v \right\rangle_{\mathcal{H}}$$

if and only if $h \in \mathcal{P}_2$ and

$$\liminf_{y \to \infty} y \, |h(iy, iy)| < \infty.$$

Hankel vector moment sequences

Let $z_Y = z_1 Y + z_2(1 - Y)$. If A is bounded then

$$h(z) = \left\langle (A - z_Y)^{-1} v, v \right\rangle = -\sum_{n=1}^{\infty} \left\langle z_Y^{-1} (A z_Y^{-1})^{n-1} v, v \right\rangle .$$

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In analogy with one variable, the function $r_n(z) = \left\langle z_Y^{-1} (A z_Y^{-1})^{n-1} v, v \right\rangle$ is called the n-th scalar moment.

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$$R_n(z) = z_Y^{-1} (A z_Y^{-1})^{n-1} v$$

is called a vector moment.

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Furthermore the vector valued expression

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These are not in general polynomials in $\frac{1}{z_1}, \frac{1}{z_2}$.

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Theorem (Agler, McCarthy)

A Pick function h has polynomial scalar moments to order $2n-1$ if and only if h has polynomial vector moments to order n.

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Using some geometry of the representation they also obtained the following.

Corollary (Agler, McCarthy)

There is a Pick function h which has prescribed polynomial scalar moments r_k *to order* $2n - 1$ *if and only if there is a rational function with those scalar moments.*

Hankel vector moment sequences

The Hankel vector moment sequence theory can be replicated at any point in $\mathbb{R}^2$. (P.)

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Using this framework, similar interpolation results can be obtained.

Theorem (P.)

Let $x_1, \ldots, x_m$ be a set of points in $\mathbb{R}^2$ in generic position. There is a Pick function h with a prescribed power series with real coefficients to odd order at each x_i if and only if there is a rational function with those prescribed expansions at each x_i.

The end

Thanks.