Injective free polynomials

J. E. Pascoe

University of California, San Diego

jpascoe@math.ucsd.edu

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The matrix universe

Let M_n be the matrices of size n.
We define the *matrix universe* M^d in d variables to be all d-tuples of matrices with the same fixed size:

$$M^d = \bigcup_{n=1}^{\infty} M_n^d.$$

Domains

A *domain* $D \subset M^d$ is a subset of the matrix universe that is closed under direct sums and similarity. That is,

- $A \in D, B \in D \Rightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in D.$
- $A \in D \Rightarrow S^{-1}AS \in D$ for every invertible matrix S.

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(We take the convention that $X = (X_i)_{i=1}^d$ and
$S^{-1}XS = (SX_iS)_{i=1}^d$)
Note the evaluation of a free polynomial P respects this structure:

- $P \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} P(A) & 0 \\ 0 & p(B) \end{pmatrix}$.
- $S^{-1}P(A)S = P(S^{-1}AS)$ for every invertible matrix S.

Derivatives of free polynomials

We fix the notation that $DP(X)[H]$ is the Gateaux derivative of the the polynomial P at X in the direction H. Thus, there is the formula for $X, H \in M_n^d$

$$DP(X)[H] = \lim_{t \to 0} \frac{P(X + tH) - P(X)}{t}.$$

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This can be taken formally via the following identities

- $Dx_i(X)[H] = H_i$,
- $D[PQ](X)[H] = (DP(X)[H])Q + P(DQ(X)[H])$, and
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Thus, the derivative of $f(x_1, x_2) = x_1 x_2$ is

$$Df(X)[H] = H_1 X_2 + X_1 H_2.$$

The Inverse Function Theorem

Theorem

Let $P : D \to M^{\tilde{d}}$ be a free polynomial map. The following are equivalent:

1. *P is injective.*
2. *$DP(X)$ is nonsingular for every matrix tuple X. That is,*

$$DP(X)[H] = 0 \text{ implies } h = 0.$$

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Thus, for free polynomials **local injectivity implies global injectivity.**

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For a given $X \in M_n$ the equation

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So, P is injective for matrices with spectrum in the right half plane, since for each X with spectrum in the right half plane, the equation $XH + HX = 0$ implies $H = 0$.

The Jacobian conjecture

Classically, many have considered the following conjecture.

Conjecture (Ott-Heinrich Keller '22)

Let $P : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. The following are equivalent:

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Much is known about injective polynomial maps. For example, Grothendieck ('66) showed they must be surjective and Rudin ('95) showed such a function's inverse is given by a polynomial. It is also known that if this result is true for cubic maps, then it is true in general due to Bass, Connell and Wright ('82).

The Jacobian conjecture (Commuting matrix version)

Commutative matrix tuples are a domain. The following is a corollary of our inverse function theorem.

Theorem

Let P be a polynomial map. The following are equivalent:

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2. *P is bijective.*
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Our result supplies the equivalence of injectivity with nonsingular derivative, the rest are previously known to be equivalent due to work on the classical Jacobian conjecture.

Special identities for free polynomials

Fact

Let P be a free polynomial, $X, Y \in M_n^d, t \in \mathbb{C}$ then

$$P \begin{pmatrix} X & t(X-Y) \\ 0 & Y \end{pmatrix} = \begin{pmatrix} P(X) & t(P(X)-P(Y)) \\ 0 & P(Y) \end{pmatrix}$$

Fact

Let P be a free polynomial. Let $X, H \in M_n^d$.

$$P \begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} P(X) & DP(X)[H] \\ 0 & P(X) \end{pmatrix}$$

Proof

We apply the following fact.

Theorem

$$f\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & Df(X)[H] \\ 0 & f(X) \end{pmatrix}$$

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If the derivative is singular at some matrix X in the direction H, then

$$f\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & Df(X)[H] \\ 0 & f(X) \end{pmatrix} = \begin{pmatrix} f(X) & 0 \\ 0 & f(X) \end{pmatrix} = f\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$$

Proof

For the converse we apply the following fact.

Theorem

$$f\begin{pmatrix} X & t(X-Y) \\ 0 & Y \end{pmatrix} = \begin{pmatrix} f(X) & t(f(X)-f(Y)) \\ 0 & f(Y) \end{pmatrix}$$

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to get

$$Df\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{bmatrix} 0 & X-Y \\ 0 & 0 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which implies $X = Y$.

The inverse function theorem

So we have proven the theorem.

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General picture

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Also, this is an algebraic theorem. It works over any field if we define

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