Matrix monotonicity in several noncommuting variables

J. E. Pascoe

University of California, San Diego

jpascoe@math.ucsd.edu

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Joint work with Ryan Tully-Doyle.

The functional calculus

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we define the expression $f(A)$ via the following formula.

$$f(A) = U^* \begin{pmatrix} f(\lambda_1) & 0 & \dots \\ 0 & f(\lambda_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} U.$$

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We say $A \leq B$ if $B - A$ is positive semidefinite. That is, $B - A \geq 0$. Given a self-adjoint matrix C diagonalized by a unitary matrix U, written as,

$$A = U^* \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} U$$

C is postive semidefinite if each $\lambda_i \geq 0$.

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We say f is **matrix monotone** if, for any natural number $n \in \mathbb{N}$, and any pair of n by n self-adjoint matrices A and B with spectrum in (a, b),

$$A \leq B \Rightarrow f(A) \leq f(B).$$

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but $\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix} - \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ 4 & 1 \end{pmatrix}$ is not positive semidefinite since the determinant is negative. Thus, x^3 is not matrix monotone on all of $\mathbb{R}$.

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Let $H = Y - X$. Consider the function $f(X + tH)$. By the fundamental theorem of calculus

$$f(Y) - f(X) = \int_0^1 \frac{d}{dt} f(X + tH) dt$$

So, it is enough to show that $\frac{d}{dt} f(X + tH)$ is postive semidefininte.

Matrix monotone functions exist

$$\begin{aligned}
\frac{d}{dt} f(X + tH) &= \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t} \\
&= \lim_{t \to 0} \frac{(-(X + tH)^{-1}) - (-X^{-1})}{t} \\
&= \lim_{t \to 0} \frac{(X + tH)^{-1}(X + tH - X)X^{-1}}{t} \\
&= \lim_{t \to 0} \frac{(X + tH)^{-1}(tH)X^{-1}}{t} \\
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Since, $H = Y - X \geq 0$, the derivative

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\frac{d}{dt} f(X+tH) = X^{-1}HX^{-1} = (H^{1/2}X^{-1})^* H^{1/2} X^{-1} \geq 0.
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So, x^{-1} is matrix monotone.

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Theorem (Löwner 1934)

Let $f : (a, b) \to \mathbb{R}$ be a bounded Borel function. The function f is matrix monotone if and only if f analytically continues to $\mathbb{H}$ as function $F : \mathbb{H} \cup (a, b) \to \overline{\mathbb{H}}$ which is continuous on $\mathbb{H} \cup (a, b)$.

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For example $x^{1/3}, \log x$ and $-\frac{1}{x}$ are matrix monotone on $(1, 2)$ but x^3 and e^x are not.

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Theorem (Agler, McCarthy, Young 2013)

Let $f : (a,b)^d \to \mathbb{R}$ be a rational function. The function f is matrix monotone when lifted to commuting tuples of matrices via the functional calculus then f analytically continues to $\mathbb{H}^d$ as function $F : \mathbb{H}^d \cup (a,b)^d \to \overline{\mathbb{H}}$ which is continuous on $\mathbb{H}^d \cup (a,b)^d$.

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The continuation of a matrix monotone rational function is in the conformal analogue of the **Schur-Agler class** for the upper half-plane, the **Löwner class**.

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We will now establish a framework for understanding power series several noncommuting variables.

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For example $(x_1 x_2)^* = x_2 x_1$.

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For example, $X^{x_1x_2} + 7X^{x_1x_2x_2x_1x_1} = X_1X_2 + 7X_1X_2X_2X_1X_1$.

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The free formal derivative of f at $X = (X_1, \dots, X_d)$ in the direction $H = (H_1, \dots, H_d)$ denoted $Df(X)[H]$ can be taken via the following relations where $x_i(X) = X_i$.

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Thus, the derivative of $f(X_1, X_2) = X_1X_2$ is

$$Df(X)[H] = H_1X_2 + X_1H_2.$$

Example of free derivative

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For self-adjoint X_1, X_2 we will rewrite derivatives in the form:

$$Df(X_1, X_2)[H_1, H_2] =$$

$$\begin{pmatrix} 1 \\ X_1 \\ X_2 \end{pmatrix}^* \begin{pmatrix} 0 & 7H_1 & 6H_1 \\ 7H_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} 1 \\ X_1 \\ X_2 \end{pmatrix}^* \begin{pmatrix} -8H_2 & 0 & 0 \\ 6H_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ X_1 \\ X_2 \end{pmatrix}.$$

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If the above power series converges on a neighborhood of $\overline{\mathbb{D}}$, classically by theorems of Nevanlinna and Löwner, it was shown that f is matrix monotone if and only if

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & a_4 & \dots \\ a_3 & a_4 & a_5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$$

Löwner's theorem in several variables at 0

For $X = (X_1, \dots X_d)$, let $\|X\| = \max_i \|X_i\|$.

Theorem (P., Tully-Doyle)

Let

$$f(X) = \sum_{I \in \mathcal{I}} c_I X^I$$

be a free power series in d variables which converges absolutely for all X such that $\|X\| < d + \epsilon$. The function f is matrix monotone on the domain of convergence, that is,

$$\forall_{1 \leq i \leq d} X_i \leq Y_i \Rightarrow f(X) \leq f(Y),$$

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$$\forall_{1 \leq i \leq d} X_i \leq Y_i \Rightarrow f(X) \leq f(Y),$$

if and only if for each n, f analytically continues as a function on d-tuples of n by n matrices over $\mathbb{C}$ with positive imaginary part (where $\mathit{Im}\ W = (W - W^*)/2i$) so that

$$\forall_{1 \leq i \leq d} \mathit{Im}\ Z_i > 0 \Rightarrow \mathit{Im}\ f(Z) > 0.$$

Coefficient condtion at 0

The Löwner theorem is proven using the following result.

Theorem (P., Tully-Doyle)

Let

$$f(X) = \sum_{I \in \mathcal{I}} c_I X^I$$

be a free power series in d variables which converges absolutely for all X such that $\|X\| < d + \epsilon$. The function f is matrix monotone on the domain of convergence if and only if each x_k-localizing matrix

$$C^k = (c_{I^* x_k J})_{I,J \in \mathcal{I}}$$

is positive semidefinite.

Models and the localizing matrices

In fact,

$$Df(X)[H] = \sum_{k=1}^{d} (X^I)^*_{I \in \mathcal{I}} (c_{I^* x_k J} H_k)_{I,J \in \mathcal{I}} (X^J)_{J \in \mathcal{I}}.$$

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implies that if each $H_k \geq 0$ then $Df(X)[H] \geq 0$.
Expressions for the second derivative similar to the above have been used by Helton and McCullough to understand free convexity and positive polynomials.

Models continued

Finally, via some relations in free analysis and by taking the square root of each C^k we obtain

$$f(X) - f(Y) = \sum_{k=1}^{d} m_Y^* \begin{array}{c} (C^k)^{1/2} \\ \otimes \\ I \end{array} \begin{array}{c} I \\ \otimes \\ X_k - Y_k \end{array} \begin{array}{c} (C^k)^{1/2} \\ \otimes \\ I \end{array} m_X,$$

which can be used to derive the analytic continuation of f via a lurking isometry argument.

Localizing matrix positivity

Let

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To show $C^k = (c_{I^* x_k J})_{I,J \in \mathcal{I}} \geq 0$, we show that for any X with $\|X\| < \frac{1}{d}$,

$$P_X C^k P_X \geq 0$$

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The V_X are finite dimensional and their union over $\|X\| < \frac{1}{d}$ is dense in $l^2(\mathcal{I})$.

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then $P_X(A - B)P_X = 0$ where P_X is the projection onto the vector space

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we merely need to find a specific C_X^k such that
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This is a finite dimensional problem since P_X is a projection onto a finite dimensional vector space.

Local argument

Let

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The derivative at X

$$Df(X) : M_n(\mathbb{C})^d \to M_n(\mathbb{C})$$

preserves the positive cone by matrix monotonicity, that is,

$$H \geq 0 \Rightarrow Df(X)[H] \geq 0.$$

Local argument II

Furthermore, the map $Df(X)$ is completely positive in each coordinate, where a linear map $L : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is **completely positive** if the extension of L to M_{nk} via the formula

$$L_k(A \otimes B) = L(A) \otimes B$$

on elements of the form $A \otimes B$ for $A \in M_n(\mathbb{C}), B \in M_k(\mathbb{C})$ and extended by linearity otherwise.

Local argument III

Theorem (Choi, Kraus)

A completely positive linear map $L : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *can be written in the form*

$$L(H) = \sum_{i=1}^{n^2} V_i^* H V_i$$

where $V_i \in M_n(\mathbb{C})$

So by the Choi-Kraus theorem,

$$Df(X)[H] = \sum_{k=1}^{d} \sum_{i=1}^{n^2} V_{i,k}^* H_k V_{i,k}$$

for some matrices $V_{i,k}$.

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for some matrices $V_{i,k}$.Via an algebraic reduction, the $V_{i,k}$ can be chosen to be the values of free polynomials $v_{i,k}$ at X.

Local argument IV

So,

$$Df(X)[H] = \sum_{k=1}^{d} \sum_{i=1}^{n^2} v_{i,k}(X)^* H_k v_{i,k}(X)$$

for some free polynomials $v_{i,k}$.

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Note $v_{i,k}(X) = \underset{I}{\overset{u_{i,k}^*}{\otimes}} \; m_X = \sum_{I \in \mathcal{I}} \overline{d_{i,k,I}} X^I$, for some $u_{i,k} = (d_{i,k,I})_{I \in \mathcal{I}} \in l^2(\mathcal{I})$.

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$$Df(X)[H] = \sum_{k=1}^{d} \sum_{i=1}^{n^2} \left(\underset{I}{\overset{u_{i,k}^*}{\otimes}} m_X \right)^* H_k \underset{I}{\overset{u_{i,k}^*}{\otimes}} m_X$$

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Note, that $P_X C^k P_X = P_X \sum_{i=1}^{n^2} u_{i,k} u_{i,k}^* P_X \geq 0$ since

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Local argument V

So,

$$Df(X)[H] = \sum_{k=1}^{d} m_X^* \begin{matrix} \sum_{i=1}^{n^2} u_{i,k} u_{i,k}^* \\ \otimes \\ H_k \end{matrix} m_X .$$

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Coefficient condition

To revisit the coefficient condition, we can relax the original theorem.

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is positive semidefinite in the sense that for each finite subset $\mathcal{J} \subset \mathcal{I}$

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$$\sqrt{1+\alpha} = 1 + \frac{1}{2}\alpha - \frac{1}{8}\alpha^2 + \frac{1}{16}\alpha^3 + \ldots,$$

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$$\begin{pmatrix} \frac{1}{2} & \frac{3}{8} \\ \frac{3}{8} & -\frac{1}{16} \end{pmatrix}$$

which is not positive semidefinite. So, f is not matrix monotone.

Matrix monotonicity in several noncommuting variables

J. E. Pascoe

University of California, San Diego

jpascoe@math.ucsd.edu

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