RESEARCH STATEMENT

J. E. PASCOE

My general research interests lie in functional analysis and its many varied applications, including matrix inequalities, moment problems, several complex variables, noncommutative function theory, multivariable operator theory, real algebraic geometry, and free probability. I will briefly describe a result I proved with Ryan Tully-Doyle that I think exemplifies my work and appeared in the *Journal of Functional Analysis* somewhat recently. After this introduction, I will more briefly describe some of my other work and possible goals.

1. Introduction

Let $\mathbb{H} \subset \mathbb{C}$ denote the complex upper half plane. That is,

$$\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}.$$

The **Pick class** is the set of analytic functions $f : \mathbb{H} \to \overline{\mathbb{H}}$. The elements of the Pick class are called **Pick functions**.

The theory of Pick functions can be used to analyze matrix monotone functions via Löwner's theorem. Given a function $f : (a, b) \to \mathbb{R}$, we extend f via the functional calculus to self-adjoint matrices A with spectrum in (a, b) by taking the diagonalization of A by a unitary matrix U, that is,

$$A = U^* \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} U,$$

and defining

$$f(A) = U^* \begin{bmatrix} f(\lambda_1) & & \\ & f(\lambda_2) & \\ & & \ddots \end{bmatrix} U. \tag{1.1}$$

A function $f : (a, b) \to \mathbb{R}$ is called **matrix monotone** if

$$A \leq B \Rightarrow f(A) \leq f(B)$$

where $A \leq B$ means that $B - A$ is positive semidefinite.

The condition that a function $f : (a, b) \to \mathbb{R}$ be matrix monotone is much stronger than that f is monotone in the ordinary sense. For example, let the function $f : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$f(x) = x^3.$$

The function f is monotone on all of the real line, $\mathbb{R}$. Note that

$$\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right),$$

since

$$\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

Date: September 17, 2017.

is a positive semidefinite matrix. However,

$$\begin{aligned} f\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 4 & 4 \\ 4 & 4 \end{smallmatrix}\right), \\ f\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 13 & 8 \\ 8 & 5 \end{smallmatrix}\right), \end{aligned}$$

and

$$\left(\begin{smallmatrix} 13 & 8 \\ 8 & 5 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 4 & 4 \\ 4 & 4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 9 & 4 \\ 4 & 1 \end{smallmatrix}\right)$$

which is not positive semidefinite since $\det \left(\begin{smallmatrix} 9 & 4 \\ 4 & 1 \end{smallmatrix}\right) = -5 < 0$. Thus, the function $f(x) = x^3$ is not matrix monotone even though it is monotone on all of $\mathbb{R}$.

In [24], Charles Löwner showed the following theorem.

Theorem 1.1 (Löwner [24])**.** *Let $f : (a, b) \to \mathbb{R}$ be a bounded Borel function. The function f is matrix monotone if and only if f is real analytic and analytically continues to the upper half plane as a function in the Pick class.*

For a modern treatment of Löwner's theorem, see e.g. [16, 10, 11].

Löwner's theorem can be used to identify whether or not many classically important functions are matrix monotone. For example, $x^{1/3}, \log x$, and $-\frac{1}{x}$ are matrix monotone on the interval $(1, 2)$, but x^3 and e^x are not.

Via the connection to moment problems and matrix monotonicity, the theory of Pick functions has deep and well-studied consequences for science and engineering. John von Neumann and Eugene Wigner applied Löwner's theorem to the theory of quantum collisions [38, 37]. Other applications include quantum data processing [6], wireless communications [23, 13] and engineering [7, 29].

With Ryan Tully-Doyle, I executed the program above in *several noncommuting variables,* in the free functional calculus. For example, we seek to understand functions of matrix variables, such as

$$f(X_1, X_2) = \sqrt{X_1X_2 + X_2X_1},$$

where they are well-defined.

As is discussed in Section 8.1 of [21], the free functional calculus is used in the study of *scalable* or *dimensionless* problems in systems engineering. Additionally, important algebraic and qualitative properties of functions are often accessible in the free case (see, e.g., [18, 14, 20]). A qualitative understanding of the free functional calculus is important for applications; it may be a way to work around the fact that *matrix calculations are computationally expensive* when analyzing matrix inqualities.

A free polynomial is much like an ordinary polynomial, except that the variables *do not commute.* For example

$$p(x_1, x_2) = 7x_1x_2^2 + x_2x_1x_2 - 8x_1^{15}$$

and

$$q(x_1, x_2) = 8x_1x_2^2 - 8x_1^{15}$$

are both free polynomials which are not equal to eachother. (That is, since $x_1x_2 \neq x_2x_1$, $x_1x_2^2 \neq x_2x_1x_2$.)

A free power series is like a normal power series in the sense that it is formula of the form

$$f(x_1, x_2, \ldots, x_d) = \sum_{w \in \mathcal{I}} a_w w(x_1, x_2, \ldots, x_d)$$

where $\mathcal{I}$ is the set of all monomials, ie free polynomials with only one term like $x_1x_2^2, x_2x_1x_2$, and x_1^{15}.

For our discussion the free functional calculus will consist of the free power series. In most cases, we will need these series to converge on some set of tuples of same-sized matrices, such as the n-tuples of contractions.

I proved the following generalization of Löwner's theorem with Ryan Tully-Doyle:

Theorem 1.2 (P., Tully-Doyle[34])**.** *Let*

$$f(X_1, \ldots, X_d) = \sum_{I \in \mathcal{I}} c_I X^I$$

be a free power series in d variables which converges absolutely for all $X = (X_1, \ldots, X_d)$ such that each $\|X_i\| < 1$. The function f is matrix monotone on the domain of convergence, that is,

$$\forall_{1 \leq i \leq d} X_i \leq Y_i \Rightarrow f(X) \leq f(Y),$$

if and only if for each n, f analytically continues as a function on d-tuples of n by n matrices over $\mathbb{C}$ with positive imaginary part (where $Im\ W = (W - W^*)/2i$) so that

$$\forall_{1 \leq i \leq d} Im\ Z_i > 0 \Rightarrow Im\ f(Z) > 0.$$

That is, a function in several noncommuting variables is matrix monotone if and only if it analytically continues to the matricial analogue of $\mathbb{H}^d$ as a noncommutative Pick function. More recently, I have generalized the above theorem to hold for significantly more general functions on domains and ranges in [33], but the spirit of the results in much the same.

2. Research and goals

One of the main **themes** of my research program is to take theorems from one variable operator theory, many of which have powerful applications in engineering, and generalize them to several variables, in the same way Löwner's theorem was generalized above. The program has two manifestations, both of which use essentially the same techniques: results in several complex variables and results in the free functional calculus described above. My other previous work includes:

Wedge-of-the-edge theorem[31]**:** The *edge-of-the-wedge theorem* is essentially an extension of the Schwarz reflection principle in several complex variables. A simple version of the edge-of-the-wedge theorem is:

Theorem 2.1 (The edge-of-the-wedge theorem[36])**.** *The is an open set D containing $\mathbb{H}^d \cup (-1,1)^d \cup -\mathbb{H}^d$ so that every function $f : \mathbb{H}^d \cup (-1,1)^d \to \mathbb{C}$ which is*

- *continuous on $\mathbb{H}^d \cup (-1,1)^d$,*
- *analytic on $\mathbb{H}^d$,*
- *and real-valued on $(-1,1)^d$.*

analytically continues to D.

I proved that a similar phenomenon holds for Pick functions, analytic functions from $\mathbb{H}^n$ into $\overline{\mathbb{H}}$ which extend continuously to a region in $\mathbb{R}^n$:

Theorem 2.2 (The wedge-of-the-edge theorem [31])**.** *There is an open set D containing 0 such that for any $\varepsilon > 0$, every function $f : \mathbb{H}^d \cup (-1,\varepsilon)^d \cup (-\varepsilon,1)^d \to \overline{\mathbb{H}}$ which is*

- *continuous on* $\mathbb{H}^d \cup (-1, \varepsilon)^d \cup (-\varepsilon, 1)^d$,
- *analytic on* $\mathbb{H}^d$,
- *and real-valued on* $(-1, \varepsilon)^d \cup (-\varepsilon, 1)^d$.

analytically continues to D.

The uniform estimates in [31] were used in [30] to relax the hypotheses of the main result of a ground-breaking paper by Agler, McCarthy and Young on Löwner's theorem in several commuting variables[3].

Jacobian conjecture for free polynomial maps[35]**:** The classical Jacobian conjecture states that a map $p : \mathbb{C}^d \to \mathbb{C}^d$ which is given by polynomials is injective if and only if the derivative, which is given by the Jacobian matrix, is nonsingular everywhere. The classical Jacobian conjecture is notoriously difficult. On the other hand, I solved a free analogue of the Jacobian conjecture. Namely, a free polynomial map is injective as a function on $\mathcal{M}_n(\mathbb{C})^d$ for all n if and only if the derivative is nonsingular everywhere on $\mathcal{M}_n(\mathbb{C})^d$ for each n.

Invariant theory for free polynomials[15]**:** In recent years, authors in combinatorics[9, 5] and functional analysis[4] have had increased interest in symmetric free polynomials. Classically, the theory of symmetric functions can be seen as a special case of invariant theory, the theory of polynomials with symmetry. With David Cushing and Ryan Tully-Doyle, I developed an invariant theory for free polynomials which is suitable for analysis. Surprisingly, out of this theory came certain trade-offs in the optimization of free polynomials. For example:

Proposition 2.3. *Let* $p(x_1, x_2)$ *be an even free polynomial. That is,*

$$p(x_1, x_2) = p(-x_1, -x_2).$$

There is a unique free polynomial $\hat{p}(u_1, u_2, u_3, u_4)$ *such that*

$$\hat{p}(x_1^2, x_1x_2, x_2x_1, x_2^2) = p(x_1, x_2).$$

Furthermore,

$$\sup_{X_1X_1^*+X_2X_2^*\leq 1} \|p(X_1, X_2)\| = \sup_{U_1U_1^*+\ldots+U_4U_4^*\leq 1} \|\hat{p}(U_1, U_2, U_3, U_4)\|$$

where the supremums range over all tuples of operators satisfying the given inequality.

A free polynomial in 4 variables can be converted to an even free polynomial in 2 variables and vice versa while still preserving the supremum of the norm. Reduction of degree in the optimization of even free polynomials is useful since it would reduce the number of matrix multiplications necessary for function evaluation, which is computationally expensive. Investigating this trade-off more thoroughly is an important research goal since it may enrich the noncommutative change of variables theory, which was thought to be quite rigid[20, 19].

My planned future work includes:

Boundary approximation for Pick functions: My thesis concerns behavior of Pick functions in several variables near the boundary. An important question along these lines is: given a Pick function, is there an (inner) rational Pick function which nicely approximates the original on $\mathbb{H}^d$ and some region in $\mathbb{R}^d$? These kind of questions are not only of theoretical interest but have real consequences of the understanding of matrix montone functions, and model-realization theory which have applications in engineering.

Recently, in my work with Kelly Bickel and Alan Sola, which will appear in *Proceedings of the London Mathematical Society*[12] we analyzed integrability of the derivatives of rational inner functions on the bidisk and the algebraic geometry of stable polynomials which we hope will shed some light on approximation theory here.

Moreover, my research program on Julia-Caratheodory type theorems has also shed some light on the local behavior of Pick functions on the distinguised boundary, and their conformal analogues, in a way that is somewhat relevant to these kinds of approximation theory questions and is also of interest from the point of view of several complex variables. Specifically, with John McCarthy, we have improved the Julia-Caratheodory theorem on the bidisk in [28], and extended it to noncommutative domains in [27]. Moreover, I have obtained higher order analogues of the Julia-Caratheodory theorem on the bi upper half plane[32] which are then amenable to rational approximation problems via the Hankel vector moment calculus developed by Agler and McCarthy in [2].

Convex functions: Kraus classically showed that matrix convex functions, have similar analytic continuation properties to those exhibited in Löwner's theorem[26]. In fact, the two theories are deeply connected in one variable. (In fact, Kraus was Löwner's PhD student[1].) For example, in Bhatia's book, *Matrix Analysis,* the topics of matrix monotone and matrix convex functions in one variable are covered in the same chapter[10]. Furthermore, Helton, McCullough and Vinnikov have described the theory of rational convex functions in the noncommmutative setting[22]. Thus, a general theory of convex functions, their analytic continuations and other qualitative properties in noncommuting variables may not be far off. In a very specific case where the domain of a matrix convex function is quite large, (which is an extremely restrictive case) we recently showed that all matrix convex functions must be actually be quadratic functions[17].

Concrete Agler model-realization theory: Most of my work depends on Agler model-realization theory. For the most part, the existence of models has been proven nonconstuctively. Recent work by Greg Knese in the commutative case for rational functions[25] and by Joseph A. Ball, Vladimir Bolotnikov and Quanlei Fang in the noncommutative case[8] suggests that a constructive theory may exist in general.

References

[1] The Mathematics Geneaology Project, 2014. `http://genealogy.math.ndsu.nodak.edu/id.php?id=153303`, accessed 9-9-2014.

[2] J. Agler and J.E. M^cCarthy. Hankel vector moment sequences and the non-tangential regularity at infinity of two variable Pick functions. *Trans. Amer. Math. Soc.*, 366(3):1379–1411, 2014.

[3] J. Agler, J.E. M^cCarthy, and N.J. Young. Operator monotone functions and Löwner functions of several variables. *Ann. of Math.*, 176:1783–1826, 2012.

[4] J. Agler and N.J. Young. Symmetric functions of two noncommuting variables. *J. Funct. Anal.*, 266(9):5709 – 5732, 2014.

[5] Marcelo Aguiar, Carlos Andr, Carolina Benedetti, Nantel Bergeron, Zhi Chen, Persi Diaconis, Anders Hendrickson, Samuel Hsiao, I. Martin Isaacs, Andrea Jedwab, Kenneth Johnson, Gizem Karaali, Aaron Lauve, Tung Le, Stephen Lewis, Huilan Li, Kay Magaard, Eric Marberg, Jean-Christophe Novelli, Amy Pang, Franco Saliola, Lenny Tevlin, Jean-Yves Thibon, Nathaniel Thiem, Vidya Venkateswaran,

C. Ryan Vinroot, Ning Yan, and Mike Zabrocki. Supercharacters, symmetric functions in noncommuting variables, and related Hopf algebras. *Adv. Math.*, 229(4):2310 − 2337, 2012.

[6] R. Ahlswede and P. Lober. Quantum data processing. *IEEE Trans. Inform. Theory*, 47(1):474–478, 2001.

[7] J.A. Alcober, I.M. Tkachenko, and M. Urrea. Construction of Solutions of the HamburgerLöwner Mixed Interpolation Problem for Nevanlinna Class Functions. In Christian Constanda and M.E. Pérez, editors, *Integral Methods in Science and Engineering, Volume 2*, pages 11–20. Birkhäuser Boston, 2010.

[8] J.A. Ball, V. Bolotnikov, and Q. Fang. Schur-class multipliers on the Fock space: de Branges-Rovnyak reproducing kernel spaces and transfer-function realizations. *J. Math. Anal. Appl.*, 341(1):519–539, 2008.

[9] N. Bergeron, C. Reutenauer, M. Rosas, and M. Zabrocki. Invariants and coinvariants of the symmetric group in noncommuting variables. *Canad. J. Math.*, 60:266 − 296, 2008.

[10] R. Bhatia. *Matrix Analysis.* Princeton University Press, Princeton, 2007.

[11] R. Bhatia. *Positive Definite Matrices.* Springer-Verlag, New York, 2007.

[12] Kelly Bickel, J. E. Pascoe, and Alan Sola. Derivatives of rational inner functions. *Proc. Lond. Math. Soc.*, 2017. to appear.

[13] Holger Boche and Eduard Axel Jorswieck. Optimization of matrix monotone functions: Saddle-point, worst case noise analysis, and applications. In *Proc. of ISIT*, 2004.

[14] J. Cimprič, J.W. Helton, I. Klep, S. McCullough, and C. Nelson. On real one-sided ideals in a free algebra. *J. Pure. Appl. Algebra*, 218(2):269 − 284, 2014.

[15] D. Cushing, J. E. Pascoe, and R. Tully-Doyle. Free functions with symmetry. *Math. Z.*, 2017. to appear.

[16] W.F. Donoghue. *Monotone matrix functions and analytic continuation.* Springer, Berlin, 1974.

[17] Bill Helton, J. E. Pascoe, Ryan Tully-Doyle, and Victor Vinnikov. Convex entire noncommutative functions are polynomials of degree two or less. *Integral Equations and Operator Theory*, 86(2):151–163, 2016.

[18] J. W. Helton. Positive noncommuative polynomials are sums of squares. *Ann. of Math.*, 156(2):675–694, 2002.

[19] J.W. Helton, I. Klep, and S. McCullough. Proper free analytic maps. *J. Funct. Anal.*, 260(5):1476–1490, 2011.

[20] J.W. Helton, I. Klep, S. McCullough, and N. Slinglend. Noncommutative ball maps. *J. Funct. Anal.*, 257:47–87, 2009.

[21] J.W. Helton and S. McCullough. Convex noncommutative polynomials have degree two or less. *SIAM J. Matrix Anal. & Appl.*, 25(4):11241139, 2004.

[22] J.W. Helton, S. McCullough, and V. Vinnikov. Noncommutative convexity arises from linear matrix inequalities. *J. Funct. Anal.*, 240(1):105–191, 2006.

[23] Eduard Axel Jorswieck and Holger Boche. *Majorization and Matrix Monotone Functions in Wireless Communications*, volume 3 of *Foundations and Trends in Communications and Information Theory.* Now publishers, July 2007. pp. 553–701.

[24] K. Löwner. Über monotone Matrixfunktionen. *Math. Z.*, 38:177–216, 1934.

[25] G. Knese. Rational inner functions in the Schur-Agler class of the polydisk. *Publ. Mat.*, 55(2):343–357, 2011.

[26] F. Kraus. Über konvexe Matrixfunktionen. *Math. Z.*, 41:18–42, 1936.

[27] John McCarthy and J. E. Pascoe. A noncommutative Julia inequality. *Math. Ann.*, 2017. to appear.

[28] John McCarthy and J. E. Pascoe. The Julia-Caratheodory theorem on the bidisk revisited. *Acta Sci. Math.*, 2017. to appear.

[29] Hiroyuki Osaka and Jun Tomiyama. Note on the structure of the spaces of matrix monotone functions. In Kalle Åström, Lars-Erik Persson, and Sergei D. Silvestrov, editors, *Analysis for Science, Engineering and Beyond*, volume 6 of *Springer Proceedings in Mathematics*, pages 319–324. Springer Berlin Heidelberg, 2012.

[30] J. E. Pascoe. Note on Löwner's theorem on matrix monotone functions in several commuting variables of Agler, McCarthy and Young. *ArXiv e-prints*, 2014. arXiv1409.2605.

[31] J. E. Pascoe. A wedge-of-the-edge theorem. *Bull. Lond. Math. Soc.*, 2017. to appear.

[32] J. E. Pascoe. An inductive Julia-Caratheodory theorem for Pick functions in two variables. *Proc. Edin. Math. Soc.*, 2017. to appear.

[33] J. E. Pascoe. The noncommutative Löwner theorem for matrix monotone functions over operator systems. submitted., 2017.

[34] J. E. Pascoe and R. Tully-Doyle. Free Pick functions: representations, asymptotic behavior and matrix monotonicity in several noncommuting variables. *J. Funct. Anal.*, 273(1):283–328, 2017.

[35] J.E. Pascoe. The inverse function theorem and the jacobian conjecture for free analysis. *Mathematische Zeitschrift*, pages 1–8, 2014.

[36] W. Rudin. *Lectures on the Edge-of-the-Wedge Theorem*. AMS, Providence, 1971.

[37] E. Wigner. On a class of analytic functions from the quantum theory of collisions. *Ann. of Math.*, 53(1):36–67, 1951.

[38] E. Wigner and J. v. Neumann. Significance of Löwner's theorem in the quantum theory of collisions. *Ann. of Math.*, 59(2):418–433, 1954.