A WEDGE-OF-THE-EDGE THEOREM: ANALYTIC CONTINUATION OF MULTIVARIABLE PICK FUNCTIONS IN AND AROUND THE BOUNDARY

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Abstract. In 1956, quantum physicist N. Bogoliubov discovered the edge-of-the-wedge theorem, a theorem used to analytically continue a function through the boundary of a domain under certain conditions. We discuss an analogous phenomenon, a wedge-of-the-edge theorem, for the boundary values of Pick functions, functions from the poly upper half plane into the half plane. We show that Pick functions which have a continuous real-valued extension to a union of two hypercubes with a certain orientation in $\mathbb{R}^d$ have good analytic continuation properties. Furthermore, we establish bounds on the behavior of this analytic continuation, which makes normal families arguments accessible on the boundary for Pick functions in several variables. Moreover, we obtain a Hartog's phenomenon type result for locally inner functions.

Contents

1. Introduction

Let Π denote the upper half plane in $\mathbb{C}$.
Consider the Schwarz reflection principle.

Theorem 1.1 (Schwarz reflection principle). *A function* $f : \Pi \cup (-1, 1) \to \mathbb{C}$ *which is*

- *continuous on* $\Pi \cup (-1, 1)$,
- *analytic on* Π,
- *and real-valued on* $(-1, 1)$,

analytically continues to $\Pi \cup (-1, 1) \cup -\Pi$.

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That is, a function on the upper half plane which has a continuous real-valued extension to $(-1,1)$ analytically continues through $(-1,1)$ to the lower half plane.

What is the generalization of the Schwarz reflection principle to several variables? Consider a function $f : \Pi^2 \cup (-1,1)^2 \to \mathbb{C}$ which is

- continuous on $\Pi^2 \cup (-1,1)^2$,
- analytic on Π^2,
- and real-valued on $(-1,1)^2$.

We would like to say that f *analytically continues to* $\Pi^2 \cup (-1,1)^2 \cup -\Pi^2$ by analogy with the reflection principle. However, $\Pi^2 \cup (-1,1)^2 \cup -\Pi^2$ is *not an open set;* so it is unclear what saying *analytically continues* would mean.

The reflection principle in several variables we will discuss is called the *edge-of-the-wedge theorem.* Rudin wrote an excellent text on the subject, called *Lectures on the edge-of-the-wedge theorem* [14].

Theorem 1.2 (The edge-of-the-wedge theorem)**.** *The is an open set D containing $\Pi^d \cup (-1,1)^d \cup -\Pi^d$ so that every function $f : \Pi^d \cup (-1,1)^d \to \mathbb{C}$ which is*

- *continuous on $\Pi^d \cup (-1,1)^d$,*
- *analytic on Π^d,*
- *and real-valued on $(-1,1)^d$,*

analytically continues to D.

1.1. **Pick functions and the wedge-of-the-edge theorem.** An analytic function

$$f : \Pi^d \to \overline{\Pi}$$

is called a Pick function.

Charles Löwner showed that a function $f : (-1,1) \to \mathbb{R}$ that is *matrix monotone*, in the sense that for any two self-adjoint matrices A, B with spectrum in $(-1,1)$,

$$A \leq B \Rightarrow f(A) \leq f(B),$$

must actually be the restriction of an analytic function on $\Pi \cup (-1,1) \cup -\Pi$ whose restriction to Π is a Pick function [11]. Löwner's theorem is valuable tool in the theory of matrix inequalities which has been applied in engineering[6, 9] and science[16, 15]. Agler, McCarthy and Young showed that an analogue of Löwner's Theorem holds for Pick functions in several variables [4].

We now give the wedge-of-the-edge theorem.

Theorem 1.3 (The wedge-of-the-edge theorem)**.** *There is an open set D containing 0 such that for any $\varepsilon > 0$, every function $f : \Pi^d \cup (-1,\varepsilon)^d \cup (-\varepsilon,1)^d \to \overline{\Pi}$ which is*

- *continuous on $\Pi^d \cup (-1,\varepsilon)^d \cup (-\varepsilon,1)^d$,*
- *analytic on Π^d,*
- *and real-valued on $(-1,\varepsilon)^d \cup (-\varepsilon,1)^d$,*

analytically continues to D.

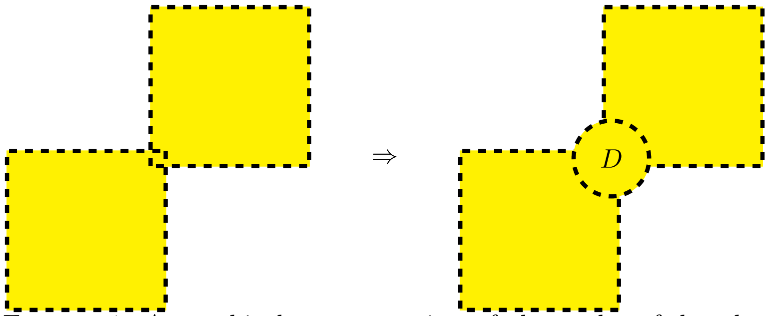

FIGURE 1. A graphical representation of the wedge-of-the-edge theorem, which appears analogous to the picture for the edge-of-the-wedge theorem. However, here we are in $\mathbb{R}^2$, which is the *edge* in the edge-of-the-wedge theorem and we are analytically continuing off the intersection of two *wedges* in $\mathbb{R}^2$.

1.2. **Some examples.** The function $\sqrt{xy}$ is defined and real on $[-1,0]^2 \cup [0,1]^2$. Also, it analytically continues to Π^2 as a Pick function. However, it cannot extend to a neighborhood of 0 because $\sqrt{xy}$ has a branch cut.

For any t, the function $\frac{x}{1-txy}$ defines a Pick function. Furthermore, each of these is real-valued on $(-\varepsilon,1)\times(-1,\varepsilon)\cup(-1,\varepsilon)\times(-\varepsilon,1)$ for small enough ε. However, for large t, the singular set of these function approach to being the x and y axes. That is, there is not a fixed set D so that all the functions in this family analytically continue to a neighborhood of 0. So, orientation matters.

1.3. **The detailed wedge-of-the-edge theorem.** We give a wedge-of-the-edge theorem with precise bounds included. When it is clear from context, we will abuse notation so that $0 = (0,0,\dots,0)$ and $1 = (1,1,\dots,1)$.

Theorem 1.4 (The detailed wedge-of-the-edge theorem)**.** *Let $f : \Pi^d \to \overline{\Pi}$ be a Pick function which has a continuous real-valued extension to $(-1,\varepsilon)^d \cup (-\varepsilon,1)^d$. For any $h \in \mathbb{C}^d$,*

$$|\frac{f^{(n)}(0)[h]}{n!}| \leq 6 \cdot 60^n \|h\|^n |f^{'}(0)[1]|.$$

(Here, $f^{(n)}(0)[h] = \frac{d^n}{dt^n} f(th)|_{t=0}$, and $\|h\| = \|h\|_\infty = \max |h_i|$.)

Namely, each f analytically continues to

$$D = \left\{ z \in \mathbb{C}^d | \|z\| < \frac{1}{60} \right\}$$

and, for all $z \in D$

$$|f(z) - f(0)| \leq \frac{360\|z\|}{1 - 60\|z\|} |f^{'}(0)[1]|.$$

Here, 60 is not a sharp constant and can probably be endlessly improved if it is necessary for a given application. In fact, it is possible to conjecture that the power series of f around zero absolutely converges on $[0,1]^d$, which would then imply continuation to all $\|z\| < 1$. However, we are essentially interested in establishing the *existence* of uniform estimates which are independent of dimension. The detailed wedge-of-the-edge is proven in Section 3.

The detailed wedge-of-the-edge theorem gives the following corollary, which makes normal families arguments accessible on the boundary for Pick functions.

Corollary 1.5. *Let $E \subseteq \mathbb{R}^d$ be a connected open set. Let $(f_n)_{n=1}^{\infty}$ be a sequence of Pick functions in d variables which have a continuous real-valued extension to E. The following are equivalent:*

1. *The sequence $(f_n)_{n=1}^{\infty}$ has a subsequence which converges uniformly on compact sets of $\Pi^d \cup E$ to a Pick function f which has a continuous real-valued extension to E.*
2. *There is a point $a \in E$ such that $f_n(a)$ is bounded and $f_n'(a)[1]$ is bounded.*

Corollary 1.5 follows from the detailed wedge-of-the-edge theorem directly, since it implies the existence of uniformly bounded analytic continuation which is then subject to classical normal families argument.

Finally, we mention a Hartog's principle type phenomenon which follows trivially from the wedge-of-the-edge theorem.

Corollary 1.6. *Let $E \subseteq \mathbb{R}^d$ be an open set. Let $p \in E$. If f is Pick function in d variables which has a continuous real-valued extension to $E \setminus \{p\}$ then, it has a continuous real-valued extension to E.*

Now, letting $\mathbb{D}^d$ denote the polydisk and $\mathbb{T}^d$ denote the distinguished boundary, we obtain that near a singular point p of a locally inner function, every contour must intersect at p. Specifically, the following corollary applies to rational inner functions and gives some idea about the geometry of their contours in the boundary.

Corollary 1.7. *Let $E \subseteq \mathbb{T}^d$ be an open set. Let $p \in E$. Let $\varphi : \mathbb{D}^d \to \mathbb{D}$ be an analytic function with a continuous unimodular extension to $E \setminus \{p\}$. Either φ has a continuous unimodular extension to E, or each contour $C_w = \{w \in E | \varphi(w) = \tau \in \mathbb{T}\}$ has p in its closure.*

2. Estimating homogeneous polynomials using their values on a hypercube

The goal of this section will be to prove the following fact which will be crucial to prove the detailed wedge-of-the-edge theorem.

Lemma 2.1. *If $p(x)$ is a homogeneous polynomial of degree n in d variables such that $|p(x)| \leq 1$ for all $x \in [0,1]^d$, then, for all $z \in \mathbb{C}^d$*

$$|p(z)| \leq (3\sqrt{2})^n \|z\|^n \frac{3^{9/4}}{64} \left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^3.$$

Namely,

$$|p(z)| \leq 6 \cdot 60^n \|z\|^n.$$

The essential part of Lemma 2.1 is the first inequality. The second inequality is meant to be a transparent estimate of the first and can be derived from it using a calculator. Since the first inequality is in the more exact form, it is far from clear what the numbers actually mean, and because our methods are somewhat naïve, it is unclear whether they are meaningful at all. To prove the lemma, we will use bounds from interpolation theory.

Let $x_0, \dots, x_n$ be distinct points in $[0,1]$ and $\lambda_0, \dots, \lambda_n \in \mathbb{C}$. There is a unique polynomial p of degree n such that $p(x_i) = \lambda_i$. Specifically, it is given by the Lagrange interpolation formula,

$$p(x) = \sum_{i=0}^{n} \lambda_i \prod_{i \neq j} \frac{x - x_j}{x_i - x_j}. \tag{2.1}$$

However, $p(x)$ may take exceptionally large values in the interval $[0,1]$, a fact known as *Runge's phenomenon.* Because of Runge's phenomenon, polynomial interpolation is often considered bad from the point of view of approximation theory. However, estimates originally derived to uncover how bad Runge's phenomenon can be will suffice for our purposes - the phenomenon is not superexponentially bad and so we will be able to use the estimates to establish the convergence of power series later on.

For example, let $x_i = \frac{i}{n}$. The Lagrange interpolation formula (2.1) becomes

$$p(x) = \sum_{i=0}^{n} \lambda_i \prod_{i \neq j} \frac{x - \frac{j}{n}}{\frac{i}{n} - \frac{j}{n}},$$

which can be simplified to

$$p(x) = \sum_{i=0}^{n} \lambda_i \frac{n^n}{n!} \binom{n}{i} \prod_{i \neq j} x - \frac{j}{n}.$$

Now, for any $z \in \mathbb{C}$,

$$|p(z)| \leq \sum_{i=0}^{n} |\lambda_i| \frac{n^n}{n!} \binom{n}{i} (|z| + 1)^n$$

$$|p(z)| \leq \max |\lambda_i| \frac{n^n}{n!} (|z| + 2)^n$$

Thus, applying the estimate, $\frac{n^n}{n!} \leq e^{n+1}$, which can be derived using the freshman calculus, we get that

$$|p(z)| \leq \max |\lambda_i| e^{n+1} (|z| + 2)^n.$$

So, we can imagine that some kind of exponential estimates as in Lemma 2.1 are plausible, with naïve calculations as above.

Interpolation is captured in terms of linear algebra by *Vandermonde matrices.* The interpolation problem can be rephased as trying to find a_i such that $\sum a_k x_i^k = \lambda_i$. That is,

$$\begin{pmatrix} 1 & x_1 & \dots & x_1^n \\ 1 & x_2 & \dots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

So, letting

$$V = \begin{pmatrix} 1 & x_1 & \dots & x_1^n \\ 1 & x_2 & \dots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{pmatrix}, a = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \lambda = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix},$$

we want to solve

$$Va = \lambda,$$

for a, which is given by the formula

$$a = V^{-1}\lambda.$$

So, ultimately, we want to find some kind of norm estimates on V^{-1} to bound a which then give bounds on the values of the polynomial.

Certain choices of $(x_i)_{i=0}^n$, the interpolation nodes, are better conditioned than others. From our perspective, good nodes are the so-called *Chebychev nodes.*

Theorem 2.2 (Gautschi [7])**.** *Let $n \in \mathbb{N}$. Let $\tilde{x}_k = \cos(\frac{2k+1}{n+1}\pi)$. Let $\tilde{V}_n$ be the Vandermonde matrix corresponding to $\tilde{x}_i$. Then,*

$$\|\tilde{V}_n^{-1}\|_\infty \leq \frac{3^{3/4}}{4(n+1)}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right].$$

So given a polynomial $p(x) = \sum_{i=0}^n a_i x^i$ such that $|p(x)| \leq 1$ on $[-1,1]$ we get that

$$|p(z)| \leq \max(1, |z|)^n \frac{3^{3/4}}{4}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right],$$

using the estimate above. Specifically, since each $\tilde{x}_i \in (-1,1)$, we know that $|f(\tilde{x}_i)| \leq 1$. So let $\lambda_i = f(\tilde{x}_i)$. Note $\|\lambda\| \leq 1$. So, since $a = \tilde{V}_n^{-1}\lambda$, we get that

$$|a_i| \leq \|\tilde{V}_n^{-1}\|_\infty \|\lambda\|_\infty \leq \frac{3^{3/4}}{4(n+1)}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right].$$

Now note that

$$|p(z)| \leq \sum_{i=0}^n |a_i||z|^i \leq (n+1)\max_i(|a_i|)\max_i(|z^i|),$$

which can then be simplified to the original claim.

To establish Lemma 2.1, we need to understand how the above argument works in *several variables.* There, however, the situation for interpolation is more dire. Let $x_1, \ldots, x_n \in \mathbb{C}^n$. Fix some corresponding multi-indices $I_1, \ldots I_n$. The linear system given by the equations

$$\sum a_I x_i^I = \lambda_i$$

for each i could be singular in some non-trivial way if all the points lie on some hidden variety with low degree defining polynomials. For example, let

$$x_1 = (1,0), x_2 = (0,1), x_3 = (-1,0).$$

Fix the multi-indices to be

$$I_1 = (0,0), I_2 = (2,0), I_3 = (0,2).$$

The linear system for interpolation is given by

$$\begin{aligned}
a_{I_1} + a_{I_2}x_1^{I_2} + a_{I_3}x_1^{I_3} &= a_{I_1} + a_{I_2} = \lambda_1 \\
a_{I_1} + a_{I_2}x_2^{I_2} + a_{I_3}x_2^{I_3} &= a_{I_1} + a_{I_3} = \lambda_2 \\
a_{I_1} + a_{I_2}x_3^{I_2} + a_{I_3}x_3^{I_3} &= a_{I_1} + a_{I_2} = \lambda_3.
\end{aligned}$$

However, the above is singular, essentially because the points lie on a circle. So, we will choose the points and the multi-indices in such a way that we avoid this problem.

However if we choose the nodes, $x_{(n_1,\ldots,n_d)} = (\tilde{x}_{n_1}, \ldots, \tilde{x}_{n_d})$ for all $(n_1, \ldots, n_d) \in \{0, \ldots, n\}^d$ where $\tilde{x}$ are as in Theorem 2.2 and the multi-indicies to be all $(n_1, \ldots, n_d) \in$

$\{0,\ldots,n\}^d$, we get that the interpolating matrix has a special form which can be nicely summed up in terms of *Kronecker products.* The Kronecker product of two matrices $A = (a_{ij})_{1\le i,j\le n}$, and $B = (b_{kl})_{1\le k,l\le m}$ is given by the block matrix $A\otimes B = (a_{ij}B)_{1\le i,j\le n}$. For example,

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \otimes \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} & 1\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} \\ 2\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} & 3\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 & 5 \\ 0 & 0 & 6 & 7 \\ 8 & 10 & 12 & 15 \\ 12 & 14 & 18 & 21 \end{pmatrix}$$

Observation 2.3. *The matrix*

$$V_n = \tilde{V}_n^{\otimes d} = \underbrace{\tilde{V}_n \otimes \ldots \otimes \tilde{V}_n}_{d\ times}$$

is the Vandermonde matrix corresponding to the nodes $x_{(n_1,\ldots,n_d)} = (\tilde{x}_{n_1},\ldots,\tilde{x}_{n_d})$ *for all* $(n_1,\ldots,n_d)\in\{0,\ldots,n\}^d$ *where* $\tilde{x}$ *are as in Theorem 2.2. and the multi-indicies to be all* $(n_1,\ldots,n_d)\in\{0,\ldots,n\}^d$. *Moreover,*

$$\|V_n^{-1}\|_\infty = \|\tilde{V}_n^{-1}\|_\infty^d.$$

For example, consider the degree one, two variable case. We have that

$$\tilde{V}_1 = \begin{pmatrix} 1 & \tilde{x}_0 \\ 1 & \tilde{x}_1 \end{pmatrix}.$$

So,

$$\tilde{V}_1 \otimes \tilde{V}_1 = \begin{pmatrix} 1 & \tilde{x}_0 & \tilde{x}_0 & \tilde{x}_0^2 \\ 1 & \tilde{x}_1 & \tilde{x}_0 & \tilde{x}_0\tilde{x}_1 \\ 1 & \tilde{x}_0 & \tilde{x}_1 & \tilde{x}_1\tilde{x}_0 \\ 1 & \tilde{x}_1 & \tilde{x}_1 & \tilde{x}_1^2 \end{pmatrix},$$

which corresponds problem of finding a polynomial with monomials $1, x_1, x_2, x_1x_2$ that interpolates certain values at the four points

$$(\tilde{x}_0,\tilde{x}_0),(\tilde{x}_1,\tilde{x}_0),(\tilde{x}_0,\tilde{x}_1),(\tilde{x}_1,\tilde{x}_1).$$

So, via the same method as in one variable, we can obtain the following.

Lemma 2.4. *Let* p *be a polynomial in three variables such that* $|p(x)|\le 1$ *for all* $x\in[-1,1]^3$. *Then,*

$$|p(z)| \le \max(1,\|z\|)^n \frac{3^{9/4}}{64}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^3.$$

Proof. Let $p(x) = \sum a_I x^I$ be a polynomial in 3 variables of degree n such that $|p(x)|\le 1$ on $[-1,1]^3$. Applying Observation 2.3 and Theorem 2.2 we get that

$$|a_I| \le \|V_n^{-1}\|_\infty \le \left[\frac{3^{3/4}}{4(n+1)}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]\right]^3.$$

Now note that

$$|p(z)| \le \sum |a_I||z|^I \le \begin{pmatrix} \#\text{ of monomials of} \\ \text{degree} \le n \end{pmatrix} \max_I(|a_I|)\max_I(|z^I|),$$

which can then be simplified to the original claim using the observation that there are less than $(n+1)^3$ monomials of degree less than or equal to n. □

Now, we will homogenize the result to obtain something like Lemma 2.1 for $d=4$.

Lemma 2.5. *Let p be a homogenous polynomial in four variables such that $|p(x)| \leq 1$ for all $x \in [0,1]^4$. Then,*

$$|p(z)| \leq 3^n \|z\|^n \frac{3^{9/4}}{64} \left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^3,$$

Proof. Let p be a homogenous polynomial in four variables such that $|p(x)| \leq 1$ for all $x \in [0,1]^4$. Namely,

$$|p(1, x_2, x_3, x_4)| \leq 1$$

for all $(x_2, x_3, x_4) \in [0,1]^3$. We will now change variables via $y_i = 2x_{i+1} - 1$. So

$$|p(1, \frac{y_1+1}{2}, \frac{y_2+1}{2}, \frac{y_2+1}{2})| \leq 1.$$

for all $(y_1, y_2, y_3) \in [-1,1]^3$. So,

$$|p(1, \frac{y_1+1}{2}, \frac{y_2+1}{2}, \frac{y_2+1}{2})| \leq \max(1, \|y\|)^n \frac{3^{9/4}}{64} \left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^3$$

for all $y \in \mathbb{C}^3$. Changing back we get that

$$|p(1, x_2, x_3, x_4)\| \leq \max(1, \|2x-1\|)^n \frac{3^{9/4}}{64} \left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^3,$$

for all $(x_2, x_3, x_4) \in \mathbb{C}^3$. Rehomogenizing, we get that

$$|p(x_1, x_2, x_3, x_4)\| \leq$$

$$\max(|x_1|, |2x_2 - x_1|, |2x_3 - x_1|, |2x_4 - x_1|)^n \frac{3^{9/4}}{64} \left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^3.$$

Note, that $\max(|x_1|, |2x_2 - x_1|, |2x_3 - x_1|, |2x_4 - x_1|) \leq 3\|x\|$, so we are done. □

Naïvely could do this for any d, to obtain a bound of

$$|p(z)| \leq 3^n \|z\|^n \left(\frac{3^{3/4}}{4}\right)^d \left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^d,$$

but we desire a uniform estimate. So instead, we use four dimensional slices to obtain better bounds.

We now prove Lemma 2.1.

Proof of Lemma 2.1. Let p be a homogeneous polynomial in d variables of degree n such that $|p(x)| \leq 1$ for $x \in [0,1]^d$. Let $z = (z_1, \ldots, z_d)$ be such that $\|z\| = 1$. (Note that it is sufficient to prove the claim for $\|z\| = 1$ since p is homogeneous.) Note z can be decomposed as $z = x^+ - x^- + iy^+ - iy^-$, where $x^+, x^-, y^+, y^- \in [0,1]^d$. Define

$$g(t_1, t_2, t_3, t_4) = p\left(\frac{t_1 x^+ + t_2 x^- + t_3 y^+ + t_4 y^-}{\sqrt{2}}\right).$$

Note that $|g(t)| \leq 1$ for all $t \in [0,1]^4$. So we get that

$$g(1, -1, i, -i) \leq 3^n \frac{3^{9/4}}{64} \left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^3$$

by Lemma 2.5. Note,

$$g(\sqrt{2}, -\sqrt{2}, i\sqrt{2}, -i\sqrt{2}) = \sqrt{2}^d g(1, -1, i, -i) = p(z).$$

Thus,

$$|p(z)| \leq (3\sqrt{2})^n \frac{3^{9/4}}{64} \left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]^3.$$

□

3. The proof of the detailed wedge-of-the-edge theorem

To continue, we will need a tool used in the proof of the original wedge-of-the-edge theorem. The following inequality was derived using the Nevanlinna representations in one [12] and two variables [5] and is a higher order analogue of the Julia inequality[13]. (The Julia inequality itself has been well-studied in one [10], two[3], and several variables[1, 2, 8].) We include a proof for completeness since it is somewhat simpler in the specific case we consider here.

Lemma 3.1 (Theorem 4.3[13])**.** *Let f be a Pick function in d variables which has a continuous real-valued extension to $(-1,\varepsilon)^d \cup (-\varepsilon,1)^d$. Then, for all $h \in [0,1]^d$*

$$|\frac{f^{(n)}(0)[h]}{n!}| \leq |f^{'}(0)[1]|.$$

Proof. Let $h \in [0,1]^d$. Note $f'(0)[h]$ is exists because the function is analytic at 0 by the edge-of-the-wedge theorem and is real. So, $f'(0)[h] = \text{Re } f'(0)[h]$. Now,

$$f'(0)[h] = \text{Re } \lim_{t\to 0} \frac{f(ith) - f(0)}{it} = \lim_{t\to 0} \frac{\text{Im } f(ith)}{t} \geq 0.$$

So, by linearity of the derivative,

$$f'(0)[1] - f'(0)[h] = f'(0)[1-h] \geq 0.$$

That is, $|f'(0)[h]| \leq |f'(0)[1]|$.

So now, it is enough show that

$$|\frac{f^{(n)}(0)[h]}{n!}| \leq |f^{'}(0)[h]|.$$

Consider the Pick function of one variable $g(z) = f(hz)$ which extends continuously to $(-1,1)$. Let $g(z)$ be given by the power series $\sum a_i z^i$ at 0. The claim is now equivalent to showing that $a_1 \geq a_n$ for all $n \geq 1$. Nevanlinna showed in [12] that there exists a measure μ on $[-1,1]$ such that

$$a_{i+1} = \int_{[-1,1]} x^i d\mu.$$

Therefore, $a_1 \geq |a_n|$ for all $n \geq 1$, so we are done. □

Now, we now note that the detailed wedge-of-the-edge theorem follows directly from the above Lemma and Lemma 2.1.

References

[1] M. Abate. The Julia-Wolff-Carathéodory theorem in polydisks. *J. Anal. Math.*, 74:275–306, 1998.

[2] M. Abate. Angular derivatives in several complex variables. In *Real methods in complex and CR geometry*, volume 1848 of *Lecture notes in Math.*, pages 1–47. Springer, Berlin, 2004.

[3] J. Agler, J.E. McCarthy, and N.J. Young. A Carathéodory theorem for the bidisk using Hilbert space methods. *Math. Ann.*, 352:581–624, 2012.

[4] J. Agler, J.E. McCarthy, and N.J. Young. Operator monotone functions and Löwner functions of several variables. *Ann. of Math.*, 176:1783–1826, 2012.

[5] J. Agler, R. Tully-Doyle, and N.J. Young. Nevanlinna representations in several variables. Submitted for publication, arXiv:1203.2261.

[6] Holger Boche and Eduard Axel Jorswieck. Optimization of matrix monotone functions: Saddle-point, worst case noise analysis, and applications. In *Proc. of ISIT*, 2004.

[7] Walter Gautschi. Norm estimates for inverses of vandermonde matrices. *Numerische Mathematik*, 23:337–347, 1974. 10.1007/BF01438260.
[8] F. Jafari. Angular derivatives in polydisks. *Indian J. Math.*, 35:197–212, 1993.
[9] Eduard Axel Jorswieck and Holger Boche. *Majorization and Matrix Monotone Functions in Wireless Communications*, volume 3 of *Foundations and Trends in Communications and Information Theory*. Now publishers, July 2007. pp. 553–701.
[10] G. Julia. Extension nouvelle d'un lemme de Schwarz. *Acta Math.*, 42:349–355, 1920.
[11] K. Löwner. Über monotone Matrixfunktionen. *Math. Z.*, 38:177–216, 1934.
[12] R. Nevanlinna. Asymptotisch Entwicklungen beschränkter Funktionen und das Stieltjessche Momentproblem. *Ann. Acad. Sci. Fenn. Ser. A*, 18, 1922.
[13] J. E. Pascoe. On the continuation of locally operator monotone functions. Unpublished, 2012.
[14] W. Rudin. *Lectures on the Edge-of-the-Wedge Theorem.* AMS, Providence, 1971.
[15] E. Wigner. On a class of analytic functions from the quantum theory of collisions. *Ann. of Math.*, 53(1):36–67, 1951.
[16] E. Wigner and J. v. Neumann. Significance of Löwner's theorem in the quantum theory of collisions. *Ann. of Math.*, 59(2):418–433, 1954.

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