Putnam Practice Problems

Practice Set 1

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1. The number 2^{29} has nine distinct digits. How can you find the missing number between 0 and 9 without using a calculator?

2. Let S be a subset of $\{1, 2, 3, \ldots, 100\}$ of size 55. Show that S has a pair of elements differing by 12. By giving an example, show that it is not necessarily true that S has a pair differing by 11.

3. (a) Let a_0, a_1, a_2, a_3 and n be integers such that a_0 is not divisible by 11 and:

   $$a_0 + a_1 n + a_2 n^2 + a_3 n^3$$

   is divisible by 11. Prove that there is an integer m such that:

   $$a_3 + a_2 m + a_1 m^2 + a_0 m^3$$

   is divisible by 11.

   (b) Show that for any prime integer p, there is a positive integer n such that:

   $$2^n + 3^n + 6^n - 1$$

   is divisible by p.

4. Show that for any n the sequence:

   $$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \quad \bmod n$$

   is eventually constant.

5. For $n \geq 3$, let a_1, a_2, $\ldots$, a_n be a sequence of prime numbers that forms an arithmetic progression. Show that the common difference is divisible by any prime p which is less than n.

6. Find all solutions of:

   $$2^x 3^y = 5^z + 1.$$

7. For a prime p, let:

   $$\frac{m}{pn} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}.$$

   Prove that $m - n$ is divisible by p^3.

8. Show that there exists an increasing sequence of positive integers $\{a_n\}_n$ such that for any non-negative integer $k \geq 0$, the sequence $\{k + a_n\}_n$ contains only finitely many primes.

9. Prove that the following system of equations have infinitely many solutions among integers:
$$\begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= l^5, \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 &= m^2, \\ x_1^5 + x_2^5 + x_3^5 + x_4^5 &= m^3. \end{cases}$$

10. A lattice point $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ in the 2-dimensional plane is called *visible* if m and n are coprime. Show that for each positive r there is a point $(k, l) \in \mathbb{Z} \times \mathbb{Z}$ such that its distance from any visible point is at least r.

11. Prove that for each positive integer n, the number
$$10^{10^{10^n}} + 10^{10^n} + 10^n - 1$$
is not prime.