Putnam practice session 10
Washington University in St. Louis — November 15, 2019

Integrals

Unless otherwise stated, evaluate the following integrals.

1. $$\int_0^\infty \frac{\ln x}{x^2+1}\,dx$$

2. $$\int_0^1 \frac{\ln(1+x)}{x^2+1}\,dx$$

3. $$\int_0^1 \frac{x-1}{\ln x}\,dx$$

4. $$\int_0^1 \frac{\ln(1-x)}{x}\,dx$$

   [Hint: Easy if you know/are allowed to use that the sum over the inverse squares of the natural numbers equals $\pi^2/6$ – hard if not, and you want to do this with "Putnam integration techniques", but possible! We'll discuss it both ways.]

5. $$\int_0^\pi \frac{x\sin(x)}{1+\cos(x)^2}\,dx$$

6. $$\int_0^1 dx \int_0^{1-x} dy\, e^{(x+y)^2}$$

7. $$\int_0^\infty \frac{dx}{\sqrt{x}}\, e^{-2019\,(x+\frac{1}{x})}$$

8. $$\int_0^a \int_0^b e^{\max(b^2x^2,a^2y^2)} dy dx,$$

   where a, b, are positive numbers.

9. (Putnam 1993) Let $K(x, y)$ be a continuous positive function on $[0, 1]^2$, and let $f(x)$, $g(x)$ be continuous positive functions on $[0, 1]$. Assume that
$$\int_0^1 f(y)K(x, y)dy = g(x), \quad \int_0^1 g(y)K(x, y)dy = f(x)$$
for all $x \in [0, 1]$. Show that then $f(x) = g(x)$ in $[0, 1]$.

10. (Putnam 2000) Show that the improper integral
$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)dx$$
converges.