Putnam Practice Session 11 **Friday, November 22**

Washington University in St. Louis, Fall 2019

Note: This week we will discuss some specific tips for the Putnam exam. Many of the problems from this worksheet come from the University of Illinois mock Putnam exams.

1. (Putnam 2001) For any real number t, let $\langle t \rangle$ denote the integer closest to t; for example, $\langle 3.14159 \rangle = 3$ and $\langle 2.71828 \rangle = 3$. Evaluate
$$\sum_{n=1}^{\infty} \frac{2^{\langle \sqrt{n} \rangle} + 2^{-\langle \sqrt{n} \rangle}}{2^n}.$$

2. Let $a_1, a_2, a_3, \ldots$ be positive real numbers, and let $S_n = \sum_{k=1}^{n} a_k$. Prove or disprove: If the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, then the series $\sum_{n=1}^{\infty} \frac{n}{S_n}$ converges as well.

3. (VTMC 2007) Let n be a positive integer, let A, B be symmetric $n \times n$ matrices with real entries. Suppose there exists $n \times n$ matrices X, Y such that $\det(AX + BY) \neq 0$. Prove that $\det(A^2 + B^2) \neq 0$.

4. Let $f(n)$ denote the n-th term in the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$, obtained by writing one 1, two 2's, three 3's, four 4's, etc.
   (a) Find, with proof, $f(2019)$.
   (b) Find, with proof, a simple general formula for $f(n)$.

5. Let $f(n)$ be the number of ordered pairs (a, b) of integers from the set $\{1, 2, \ldots, n\}$ such that $a + b$ is a perfect square (i.e., of the form k^2 for some integer k). Prove that the limit $\lim_{n\to\infty} f(n)n^{-3/2}$ exists and express this limit in the form $r(\sqrt{s} - t)$, where s and t are integers and r is a rational number.

6. (Putnam 1980) For which real numbers c do we have
$$\frac{1}{2}(e^x + e^{-x}) \leq e^{cx^2}$$
for all real numbers x? Prove your answer.

7. The sequence of digits
$$1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \ldots$$
is obtained by writing out the natural numbers in order. Let $f(n)$ denote the position of the first digit of the number n in this sequence. For example, $f(1) = 1$, $f(2) = 2$, $f(11) = 12$, $f(13) = 16$. Find, with proof, a simple explicit formula for $f(10^k)$ where k is an arbitrary positive integer.

8. (Putnam 1997) Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

9. (Putnam 2007) A triangulation $\mathcal{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side of P is a side of exactly one triangle in $\mathcal{T}$. Say that $\mathcal{T}$ is admissible if every internal vertex is shared by 6 or more triangles. Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.

10. Given an integer $n \geq 2$, let $f(n)$ denote the number of ordered pairs of non-empty, disjoint subsets of an n-element set. Find a simple formula for $f(n)$.