Putnam practice session 2
Washington University in St. Louis — Fri, September 13, 2019

Inequalities; aspects of invariants

Before working on these problems, familiarize yourself with basic inequalities, in particular the Cauchy-Schwarz, Triangle, Bernoulli, and the (Root-Mean Square)-(Arithmetic Mean)-(Geometric Mean)-(Harmonic Mean) inequality.

1. For positive numbers x, y, z such that $x+y+z=1$, find the minimum value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

2. Show that $n! < (\frac{n+1}{2})^n$ for $n = 2, 3 \ldots.$

3. Show that
$$2^x + 3^x - 4^x + 6^x - 9^x \leq 1$$
for all real x.

4. Let x, y be real and $0 < x, y < 1$. Show that $x^y + y^x > 1$.

5. Let $x_1, \ldots, x_n$ be n given real numbers. For a permutation σ of n objects, let
$$M(\sigma) = x_1 x_{\sigma_1} + x_2 x_{\sigma_2} + \ldots + x_n x_{\sigma_n}.$$
Find the maximum value of M_σ over all permutations.

6. Find all positive integers $n, a_1, \ldots, a_n$ such that $\sum_{k=1}^n a_k = 5n - 4$ and $\sum_{k=1}^n a_k^{-1} = 1$.

7. Let $a_1, \ldots, a_n$ be a sequence of positive integers and σ a permutation of n objects. Show that
$$\frac{a_1}{a_{\sigma_1}} + \ldots \frac{a_n}{a_{\sigma_n}} \geq n\,.$$

8. For real numbers $a > b > 0$, show that $\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}$.

9. If a, b, c are sides of a triangle, show that
$$\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} \geq 3\,.$$

10. Out of boredom, you decide to play the following game. Consider the infinite square lattice in the plane, given by points with integer coordinates (i, j). You start by placing one coin at the origin. You may now execute moves where you remove one coin and occupy precisely two of its neighboring lattice points with coins, which must both be unoccupied prior to the move. Your goal is to clear the square containing lattice points with $|i|, |j| \leq 5$ of all coins. Can you succeed?

    Hint: Consider the quantity $\sum_{(i,j)} \frac{1}{2^{|i|+|j|}}$, where the sum goes over all occupied lattice points.

11. (a) Consider a decomposition of 3-dimensional space into c distinct components by means of a finite number of pairwise distinct planes. If three or more planes intersect in precisely one point, we call such a point a "vertex". Let v be the number of such vertices. The intersection lines between any two planes my be cut into separate components by vertices. We call any such component an "edge". Let e be the total number of edges. Furthermore, any plane may be cut into separate components by edges. We call such components "faces" and let f be the total number of faces. Prove that

    $$v - e + f - c = -1 \; .$$

    (b) Consider a regular dodecahedron. Its faces lie in twelve distinct planes, which cut 3-dimensional space into c components. Determine c.

12. Let n be a positive integer, and $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ non-negative real numbers. Prove that

    $$(a_1 \cdot \ldots \cdot a_n)^{1/n} + (b_1 \cdot \ldots \cdot b_n)^{1/n} \leq ((a_1 + b_1) \cdot \ldots \cdot (a_n + b_n))^{1/n} \; .$$

13. Let n be a positive integer, and $a_1, \ldots, a_n$ non-negative real numbers. Prove that

    $$\frac{a_1^2}{a_1 + a_2} + \frac{a_2^2}{a_2 + a_3} + \ldots \frac{a_n^2}{a_n + a_1} \geq \frac{a_1 + a_2 + \ldots + a_n}{2} \; .$$