Putnam Practice Session 3 **Friday, September 20**

Washington University in St. Louis, Fall 2019

1. Let
$$f(n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k+1}}.$$
Evaluate $f(9999)$.

2. Take a walk on the number line, starting at 0, by on the first step taking a step of length one, either right or left, on the second step taking a step of length two, either right or left, and in general on the k-th step taking a step of length k, either right or left.

   (a) Prove that for each integer m, there is a walk that visits (has a step ending at) m.

   (b) Let $f(m)$ denote the least number of steps needed to reach m. Show that
   $$\lim_{m\to\infty} \frac{f(m)}{\sqrt{m}}$$
   exists, and find the value of the limit.

3. Let a_n be the number of different ways of covering a 1 by n strip with 1 by 1 and 1 by 3 tiles. Prove that $a_n < (1.5)^n$.

4. Define a sequence $\{a_n\}_{n\geq 1}$ by
$$a_1 = 1, \quad a_{2n} = a_n, \quad \text{and} \quad a_{2n+1} = a_n + 1.$$
Prove that a_n counts the number of 1's in the binary representation of n.

5. The numbers 1 through $2n$ are partitioned into two sets A and B of size n, in an arbitrary manner. The elements $a_1, \ldots, a_n$ of A are sorted in increasing order, that is, $a_1 < a_2 < \cdots < a_n$, while the elements $b_1, \ldots, b_n$ of B are sorted in decreasing order, that is, $b_1 > b_2 > \cdots > b_n$. First (with proof) the value of the sum
$$\sum_{i=1}^{n} |a_i - b_i|.$$

6. Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s,n) = 1$ or $\gcd(s,n) = s$. Show that there exist $s,t \in S$ such that $gcd(s,t)$ is prime.

7. The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

8. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B with are the vertices of an equilateral triangle.

9. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

10. Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a \mid a \in S_0\}$.