Putnam practice session 4
Washington University in St. Louis — September 28, 2019

Analysis

Themes include limits, (Riemann) sums, recurrences and functional equations.

1. Let $f(x)$ be an infinitely differentiable real function defined on some open interval containing $[0,1]$. For $n = 1,2,3,\ldots$ let f satisfy $f(\frac{1}{n}) = n^2/(1+n^2)$. From this determine all derivatives of f at $x = 0$.

2. Let $f : [-1,1] \to \mathbb{R}$ be a continuous function satisfying

   (a) $f(x) = \frac{2-x^2}{2} f(\frac{x^2}{2-x^2})$ for all $x \in [-1,1]$,

   (b) $f(0) = 1$, and

   (c) $\lim_{x\to 1^-} f(x)/\sqrt{1-x}$ exists and is finite.

   Prove that $f(x)$ exists, is unique, and determine $f(x)$ in closed form.

3. For every positive integer n, show that

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

4. For z in the subset $D = \{z \in \mathbb{C} : |z| \le 1, z \neq 1\}$ of the complex plane, consider

$$s(z) = \sum_{n=1}^{\infty} z^n/n\,.$$

   Show that in D, the right hand side converges and defines a continuous function.

5. Let q be an integer with $q \equiv 3 \mod 4$. Show that for any positive integer n the number $\sum_{k=0}^{\infty}(-1)^k\binom{n}{2k}q^k$ is divisible by 2^{n-1}.

6. Calculate

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m(n3^m + m3^n)}\,.$$

7. Let f be a continuous and monotonously increasing function on $[0,1]$ with $f(0)=0$ and $f(1)=1$ and inverse function f^{-1}. Prove that

   $$f(0.1)+f(0.2)+\ldots+f(0.9)+f^{-1}(0.1)+f^{-1}(0.2)+\ldots+f^{-1}(0.9)\leq 9.9\,.$$

8. For x with $|x|\neq 1$ calculate

   $$\sum_{n=0}^{\infty}\frac{x^{(2^n)}}{1-x^{(2^{n+1})}}\,.$$

9. Evaluate

   $$\log_2\left(\prod_{a=1}^{2015}\prod_{b=1}^{2015}(1+e^{\frac{2\pi iab}{2015}})\right)\,.$$

10. Evaluate

    $$\lim_{n\to\infty}\frac{1}{n^4}\prod_{j=1}^{2n}\left(n^2+j^2\right)^{1/n}\,.$$

11. Let $\sum_{n=1}^{\infty}a_n$ be a convergent series of positive real numbers. Prove that $\sum_{n=1}^{\infty}a_n^{n/(n+1)}$ also converges.

12. Determine all real functions f defined on $\mathbb{R}\setminus\{-\frac{1}{3},\frac{1}{3}\}$ that satisfy

    $$f(\frac{x+1}{1-3x})+f(x)=x\,.$$