Putnam Practice Session 5 **Friday, October 4**

Washington University in St. Louis, Fall 2019

Note: This week's problem set has two themes: the pigeonhole principle and combinatorics.

1. (Putnam 2006) Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (Here's an example of a winning strategy: If $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

2. 10 points are placed randomly on a 1×1 square. Show that there must be some pair of points that are within distance $\sqrt{2}/3$ of each other.

3. Given a sequence $a_1, \ldots, a_m$ of length m, show that there is a consecutive subsequence whose sum is divisible by m. (A consecutive subsequence means a subsequence $a_i, a_{i+1}, a_{i+1}, \ldots, a_{i+j-1}$ of length j, where j could be as small as one.)

4. Show that for any set of five integers, we can always choose three of these integers whose sum is a multiple of 3.

5. (Putnam 2010) There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

6. (a) (This is often called *Posá's soup problem.*) 51 different integers are chosen between 1 and 100, inclusive. Show that some two of them are coprime (have no prime factor in common).

   (b) 51 different integers are chosen between 1 and 100, inclusive. Show that there are some two of them such that one divides the other.

7. (Putnam 2003) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$. For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

8. (Harvey Mudd Putnam prep class) A house has one entrance and many rooms. Every room has 1, 2 or 4 doors, and these doors lead directly to other rooms or to the outside. The rooms with 1 door are precisely the bathrooms. Prove that this house must have an odd number of bathrooms.

9. (Putnam 2006) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

10. (Putnam 2013) Define a function $w : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|, |b| < 2$, let $w(a, b)$ be as in the table shown; otherwise, let $w(a, b) = 0$.

		b				
		-2	-1	0	1	2
	-2	-1	-2	2	-2	-1
	-1	-2	4	-4	4	-2
a	0	2	-4	12	-4	2
	1	-2	4	-4	4	-2
	2	-1	-2	2	-2	-1

For every finite subset S of $\mathbb{Z} \times \mathbb{Z}$, define

$$A(S) = \sum_{(s,s') \in S \times S} w(s - s').$$

Prove that if S is an finite nonempty subset of $\mathbb{Z} \times \mathbb{Z}$, then $A(S) > 0$. For example, if

$$S = \{(0, 1), (0, 2), (2, 0), (3, 1)\},$$

then the terms in $A(S)$ are $12, 12, 12, 12, 4, 4, 0, 0, 0, 0, -1, -1, -2, -2, -4, -4$.