Putnam Practice Problems

Practice Set 6

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1. Suppose that (x_1, y_1), (x_2, y_2) and (x_3, y_3) are three points on the parabola $y^2 = ax$ which have the property that the three lines orthogonal to the parabola at these points are concurrent. Prove that $y_1 + y_2 + y_3 = 0$.

2. (a) In a triangle ABC, points D and E belong to the sides BC and AC such that
$$\frac{BD}{DC} = 3 \qquad \frac{AE}{EC} = \frac{3}{2}.$$
Let P denote the intersection point of AD and BE. Find the ratio between the lengths of BP and PE.

   (b) In a triangle ABC, points E and F belong to the sides AC and AB such that
$$\frac{AE}{EC} = 4 \qquad \frac{AF}{FB} = 1.$$
Suppose D is a point that belongs to BC and Q denotes the intersection point of AD and EF. If $\frac{AG}{GD} = \frac{3}{2}$, then find the ratio between the lengths of BD and DC.

3. Given a point P on the circumference of a circle C and the vertices A_1, A_2, $\ldots$, A_n of a regular polygon inscribed in C prove that
$$PA_1^4 + PA_2^4 + \cdots + PA_n^4$$
is independent of the position of P.

4. Find all maps $u : \mathbb{R} \to \mathbb{R}^3$ satisfying the differential function
$$u(t) \times u'(t) = v(t)$$
where $v : \mathbb{R} \to \mathbb{R}^3$ is a twice differentiable map such that $v(t)$ and $v'(t)$ are never zero or parallel.

5. Let P be a point on the hyperbola $xy = 4$, and Q a point on the ellipse $x^2 + 4y^2 = 4$. Prove that the distance from P to Q is greater than 1.

6. Prove that the four lines through the centroids of the four faces of a tetrahedron perpendicular to those faces are concurrent if and only if the four altitudes of the tetrahedron are concurrent.

7. Let $ABCDEF$ be a hexagon inscribed in a circle of radius r. Show that if $AB = CD = EF = r$, then the midpoints of BC, DE, and FA are the vertices of an equilateral triangle.

8. Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = 0$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in G$.

9. For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \leq i < j < k \leq m)$ is said to be *area definite* for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\Delta A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

10. Let ABC be a triangle. The triangles PAB and QAC are constructed outside of the triangle ABC such that $AP = AB$, $AQ = AC$, and $\angle BAP = \angle CAQ$. The segments BQ and CP meet at R. Let O be the circumcenter of the triangle BCR. Prove that AO and PQ are orthogonal.