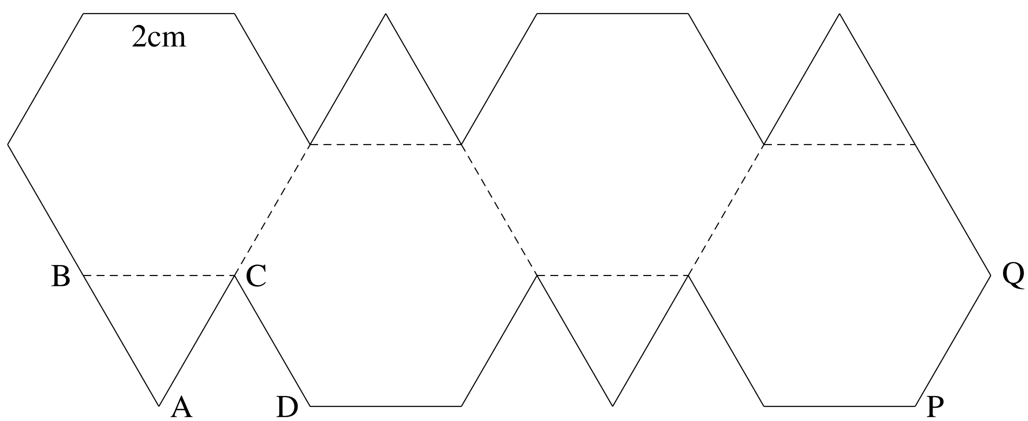
Putnam practice #7

(note: This is an original VTRMC exam)

1. Find all integers n for which $n^4+6n^3+11n^2+3n+31$ is a perfect square.

2. The planar diagram below, with equilateral triangles and regular hexagons, sides length 2cm., is folded along the dashed edges of the polygons, to create a closed surface in three dimensional Euclidean spaces. Edges on the periphery of the planar diagram are identified (or glued) with precisely one other edge on the periphery in a natural way. Thus for example, BA will be joined to QP and AC will be joined to DC. Find the volume of the three-dimensional region enclosed by the resulting surface.



3. Let $(a_i)_{1\le i\le 2015}$ be a sequence consisting of 2015 integers, and let $(k_i)_{1\le i\le 2015}$ be a sequence of 2015 positive integers (positive integer excludes 0). Let

$$A = \begin{pmatrix} a_1^{k_1} & a_1^{k_2} & \cdots & a_1^{k_{2015}} \\ a_2^{k_1} & a_2^{k_2} & \cdots & a_2^{k_{2015}} \\ \vdots & \vdots & \cdots & \vdots \\ a_{2015}^{k_1} & a_{2015}^{k_2} & \cdots & a_{2015}^{k_{2015}} \end{pmatrix}.$$

Prove that 2015! divides $\det A$.

4. Consider the harmonic series $\sum\limits_{n\geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} \cdots$. Prove that every positive rational number can be obtained as an *unordered* partial sum of this series. (An unordered partial sum may skip some of the terms $\frac{1}{k}$.)

5. Evaluate $\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} \, dx$ (where $0 \leq \arctan(x) < \pi/2$ for $0 \leq x < \infty$).

6. Let $(a_1, b_1), \ldots, (a_n, b_n)$ be n points in $\mathbb{R}^2$ (where $\mathbb{R}$ denotes the real numbers), and let $\varepsilon > 0$ be a positive number. Can we find a real-valued function $f(x,y)$ that satisfies the following three conditions?

   (a) $f(0,0) = 1$;

   (b) $f(x,y) \neq 0$ for only finitely many $(x,y) \in \mathbb{R}^2$;

   (c) $\sum\limits_{r=1}^{r=n} |f(x + a_r, y + b_r) - f(x,y)| < \varepsilon$ for every $(x,y) \in \mathbb{R}^2$.

   Justify your answer.

7. Let n be a positive integer and let $x_1, \ldots, x_n$ be n nonzero points in $\mathbb{R}^2$. Suppose $\langle x_i, x_j \rangle$ (scalar or dot product) is a rational number for all i, j ($1 \leq i, j \leq n$). Let S denote all points of $\mathbb{R}^2$ of the form $\sum_{i=1}^{i=n} a_i x_i$ where the a_i are integers. A closed disk of radius R and center P is the set of points at distance at most R from P (includes the points distance R from P). Prove that there exists a positive number R and closed disks $D_1, D_2, \ldots$ of radius R such that

   (a) Each disk contains exactly two points of S;

   (b) Every point of S lies in at least one disk;

   (c) Two distinct disks intersect in at most one point.