Putnam Practice Problems

Practice Set 8

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1. Suppose G is a group and $a, b \in G$ satisfy the relations:

$$aba^{-1} = b^{-1}, \qquad bab^{-1} = a^{-1}.$$

   Prove that $a^4 = b^4 = 1$.

2. Let G be a set with an associative binary operation $*$ such that it satisfies the relation:

$$a * a * b = b = b * a * a$$

   for all a, $b \in G$. Show that $(G, *)$ is a commutative group.

3. Let R be a ring with identity and a be an element of R such that there is a unique b satisfying $ab = 1$. Prove that $ba = 1$.

4. Suppose that H is a subgroup of a group G with size h and $a \in G$ such that for any $h \in H$ we have $(ah)^3 = 1$. Prove that the set of elements of the form:

$$ah_1ah_2 \dots ah_n$$

   for $h_i \in H$ and n a positive integer consist of at most $3h^2$ elements.

5. Let S be the smallest set of rational functions containing $f(x, y) = x$ and $g(x, y) = y$ and closed under subtraction and taking reciprocals. Show that S does not contain the nonzero constant functions.

6. Let G be a group with the following properties:

   (i) G has no element of order 2;

   (ii) $(xy)^2 = (yx)^2$, for all $x, y \in G$.

   Prove that G is Abelian.

7. Let G be a finite multiplicative group of matrices with complex entries. If M is the sum of the elements of G show that $\det(M)$ is an integer.

8. Let x and y be elements in a ring with identity and n a positive integer. Prove that if $1 - (xy)^n$ is invertible, then so is $1 - (yx)^n$.

9. Let $*$ be a binary operation on the set $\mathbb{Q}$ of rational numbers that is associative and commutative and satisfies $0 * 0 = 0$ and $(a + c) * (b + c) = a * b + c$ for all $a, b, c \in \mathbb{Q}$. Prove that either $a*b = \max(a, b)$ for all $a, b \in \mathbb{Q}$, or $a*b = \min(a, b)$ for all $a, b \in \mathbb{Q}$.

10. Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_r}h^{n_r}$$

with $1 \le r \le |G|$ and $m_1, n_1, m_2, n_2, \ldots, m_r, n_r \in \{-1, 1\}$. (Here $|G|$ is the number of elements of G.)

11. Let m and n be positive integers with $\gcd(m, n) = 1$, and let

$$a_k = \left\lfloor \frac{mk}{n} \right\rfloor - \left\lfloor \frac{m(k-1)}{n} \right\rfloor$$

for $k = 1, 2, \ldots, n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1}gh^{a_2}\cdots gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$. (As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

12. Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = 0$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in G$.