Putnam Practice Session 9 **Friday, November 8**

Washington University in St. Louis, Fall 2019

Note: This week's problem set has two themes: polynomials and games.

1. (Harvey Mudd Putnam Prep class) For which real values of p and q are the roots of the polynomial $x^3 - px^2 + 11x - q$ three consecutive integers? Give the roots in these cases.

2. (Putnam 1971)

   (a) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x+1) = p(x) + 1$ for all x.

   (b) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x^2+1) = (p(x))^2 + 1$ for all x.

3. (Putnam 1990) Is there a sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for each $n = 1, 2, 3, \ldots$ the polynomial $p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$ has exactly n distinct real roots?

4. A locker room has 100 lockers, numbered 1 to 100, all closed. I run through the locker room, and open every locker. Then I run through the room again, and close the lockers numbered $2, 4, 6, \ldots$ (all the even numbered lockers). Next I run through the room, and change the status of the lockers numbered $3, 6, 9, \ldots$ (opening the closed ones, and closing the open ones). I keep going in this manner (on the i-th run through the room, I change the status of the lockers numbered $i, 2i, 3i, \ldots$), until on my 100-th run through the room I change the status of locker 100 only. At the end of all this, which lockers are open?

5. (Putnam 2008) Alan and Barbara play a game in which they take turns filling entries of an initially empty 1024 by 1024 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

6. (Putnam 1999) A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, then the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with a total cost of exactly c?

7. (Putnam 2005) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: ($\lfloor r \rfloor$ is the greatest integer less than or equal to $r \in \mathbb{R}$.)

8. (Putnam 2002) Consider a convex polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

   Each player, in turn, signs their name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

   Show that the player who signs the first will always win by playing as well as possible.

9. (Putnam 2014) Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

10. (University of Texas Putnam prep) Two players play a game in which the first player places a king on an empty 8 by 8 chessboard, and then, starting with the second player, they alternate moving the king (in accordance with the rules of chess) to a square that has not been previously occupied. The player who moves last wins. Which player has a winning strategy?