Washington University in St. Louis, Fall 2020

Instructions: Welcome! The goal today is to discuss some (or all) of the following problems. This won't be much fun until you try and solve some of them yourself. Therefore, we will begin by dividing into breakout rooms for 30 min to work in groups as follows:

- Breakout Room 1: Problems 1-2
- Breakout Room 2: Problems 3-5
- Breakout Room 3: Problems 6-8
- Breakout Room 4: Problems 9-10

After 30 min we will reconvene, and break into groups again– this time you can choose a new subset of problems to work on, or decide to keep working on the same ones. Finally, we will end our problem solving session together with students presenting solutions to the group. I can also give some hints regarding any remaining unsolved problems.

Problems:

1. Alice and Bob want to know Carole's birthday. She tells them that it is one of ten dates, shown in the table below.

May		15	16			19
June				17	18	
July	14		16			
August	14	15		17		

   Then she whispers the *month* of her birthday to Alice, and the *day* of her birthday to Bob. The following conversation ensues:

   - **Alice:** Bob, I know that you don't know Carole's birthday.
   - **Bob:** Now I know!
   - **Alice:** And now so do I!

   What is Carole's birthday?

2. Suppose you are given three cups– one is upside down and the other two are right side up. Can you turn all cups right side up in six moves? You must turn exactly two cups over in each move.

3. Suppose that a regular octagon is tiled with non-overlapping parallelograms. Show that at least 2 of the parallelograms are rectangles.

4. (Putnam 1996) Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

5. (Putnam 2008) What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

6. (Putnam 2002) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

7. (1972 International Mathematical Olympiad) Prove that from a set of ten distinct two-digit numbers, it is possible to select two nonempty disjoint subsets whose members have the same sum.

8. (Putnam 2013) A regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

9. We place $4n$ points on a circle. Then we paint any $2n$ of them in red and the other $2n$ of them in blue. Prove that regardless of which points we have painted each color, there is always a straight line dividing the circle in two parts leaving exactly n red points on one side and n blue points on the other.

10. During a particularly boring Zoom lecture, each of five participants fell asleep exactly twice. For each pair of these five, there was some moment when both were sleeping simultaneously. Prove that at some point, at least three of them were sleeping simultaneously.