Putnam Practice Problems

Practice Set 2

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1. Find all functions $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ which satisfy

   (i) $f(m, m) = m$;

   (ii) $f(m, n) = f(n, m)$;

   (iii) $f(m, n) = f(n, m + n)$.

2. We assign a positive integer to each point with integer coordinates in the plane such that the value at each point is equal to the arithmetic mean of its four neighbors. Find all such assignments.

3. (a) Show that there are infinitely many integers n such that $3^n - 1$ is divisible by n.

   (b) (Putnam 1939) Show that there is no integer $n > 1$ such that $2^n - 1$ is divisible by n.

4. Show that for any n the sequence:

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \mod n$$

   is eventually constant.

5. (a) Show that an integer of the form $m^2 + n^2$ is not divisible by a prime of the form $4k + 3$.

   (b) Show that the equation $a^2 = b^3 + 7$ does not have a solution among integers.

6. Show that there is no polynomial $p(x)$ with integer coefficients such that $p(11) = 1$ and $p(19) = 5$.

7. (Putnam 2008) Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2 - 1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3 - 1)$ are distinct modulo p^3.

8. (Putnam 2014) A *base* 10 *over-expansion* of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

   with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \ldots, 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

9. Prove that there exists a set of positive integers A such that for any infinite set of primes S there are $m \in A$ and $n \notin A$ such that both of m and n are products of k distinct elements of S for some $k \geq 2$.

10. Show that there exists an increasing sequence of positive integers $\{a_n\}_n$ such that for any non-negative integer $k \geq 0$, the sequence $\{k + a_n\}_n$ contains only finitely many primes.