Putnam practice session 3
Washington University in St. Louis Fri, October 9, 2020

Inequalities

Before working on these problems, familiarize yourself with basic inequalities, in particular the AM-GM-HM inequality (and its weighted versions!), as well as Jensen's and the Cauchy-Schwarz, Triangle, and Bernoulli inequalities.

1. For positive numbers a, b, c with $abc = 1$, show that
$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9\,.$$

2. For positive numbers $x_1, \ldots, x_n$, what is the minimum value of
$$\frac{x_1}{\sum_i x_i - x_1} + \ldots + \frac{x_n}{\sum_i x_i - x_n} \quad ?$$

3. (Polish Math Olympiad) For a given integer $n \geq 1$, compute the minimum value of the sum
$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \ldots + \frac{x_n^n}{n}\,,$$
where the $x_1, \ldots, x_n$ are positive numbers satisfying
$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = n\,.$$

4. For a polynomial $P(x)$ with positive coefficients, show that
$$P(\frac{1}{x}) \geq \frac{1}{P(x)}\,.$$
holds for all $x > 0$ if holds for $x = 1$.

5. (Putnam) Find all positive integers $n, a_1, \ldots, a_n$ such that $\sum_{k=1}^n a_k = 5n - 4$ and $\sum_{k=1}^n a_k^{-1} = 1$.

6. (IMO) For positive a, b, c with $abc = 1$, show that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}\,.$$

7. (Putnam) Let $x_i \in (0, \pi)$, $i = 1, 2, \ldots n$, and let x be their arithmetic mean. Show that
$$\prod_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \left(\frac{\sin x}{x} \right)^n .$$

8. (A named inequality) Let $a_n > 0$ with $\sum_n a_n$ convergent. Show that
$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n .$$

9. (Another named inequality) For $p > 1$ and $a_1, a_2, \ldots, a_n$ positive, show that
$$\sum_{k=1}^{n} \left(\frac{a_1 + a_2 + \ldots + a_k}{k} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{k=1}^{n} a_k^p .$$

10. (Putnam) Let $p(z)$ be a polynomial of degree n with real coefficients whose roots all lie inside a circular disk of radius R in the complex plane. Show that for any real k, there is a circular disk of radius $R + |k|$ in the complex plane containing all the zeros of the polynomial $np(z) - kp'(z)$. Here, $p'(z)$ is the derivative of $p(z)$.

11. (French Math Olympiad) Let x, y be real and $0 < x, y < 1$. Show that $x^y + y^x > 1$.

12. (Putnam) Let f be a continuous function on the unit square. Show that
$$\int_0^1 \left(\int_0^1 f(x,y) dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y) dy \right)^2 dx \leq$$
$$\left(\int_0^1 \int_0^1 f(x,y) dx dy \right)^2 + \int_0^1 \int_0^1 f(x,y)^2 dx dy .$$