Putnam Practice Session 4 **Friday, October 16**

Washington University in St. Louis, Fall 2020

Instructions: Welcome! The goal today is to discuss some (or all) of the following problems; our theme is *linear algebra*. To start, you should review the determinant of a matrix and the definition of eigenvectors and eigenvalues– as these things play a central role in many (but not all) of the problems below. We will begin by dividing into breakout rooms to work in groups as follows:

- Breakout Room 1: Problems 1-2
- Breakout Room 2: Problems 3-4
- Breakout Room 3: Problems 5-7
- Breakout Room 4: Problems 8-11.

After discussing their problems, each group can choose any other problems to work on. Finally, we will end our problem solving session together with students presenting solutions to the group.

Problems involving some linear algebra:

1. (Northwestern Putnam Prep) Two players take turns filling out blanks in the following system of equations with real numbers.

$$\begin{cases} x + \underline{\quad}\, y + \underline{\quad}\, z = \underline{\quad} \\ x + \underline{\quad}\, y + \underline{\quad}\, z = \underline{\quad} \\ x + \underline{\quad}\, y + \underline{\quad}\, z = \underline{\quad} \end{cases}$$

Show that the first player can produce an inconsistent system no matter what the second player does.

2. (Putnam) Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n-1) \times (2^n-1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

3. Suppose there is a town with n residents, who love forming different clubs. To limit the number of possible clubs, the town council establishes the following rules:

   - Every club must have an odd number of members.
   - Any two clubs must share an even number of members.

   Show that the number of clubs is less than or equal to n.

4. (Putnam) For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, and every dot product of two different rows is odd?

Problems involving more abstract linear algebra:

5. Suppose we have 13 integers with the following property: if we remove any one of the numbers, the remaining 12 can be split into two sets of 6 numbers each with equal sum. Prove that all 13 numbers are equal. (Challenge: Can you prove the same result for 13 real numbers?)

6. In a palace, there are 9 rooms arranged in a square–3 rows of rooms with 3 rooms in each row. In every room there is a light switch which not only switches on/off the light in that room, but also switches the lights in the adjacent rooms–the room to the right, to the left, the room above and the room below. Initially, all of the lights are turned off. Is it possible to turn on all the lights in every room?

7. (MIT Putnam Prep) Let $M(n)$ denote the space of all real $n \times n$ matrices. Thus $M(n)$ is a real vector space of dimension n^2. Let $f(n)$ denote the maximum dimension of a subspace V of $M(n)$ such that every nonzero element of V is invertible.

   (a) Show that $f(n) \leq n$.

   (b) Show that if n is odd, then $f(n) = 1$.

Miscellaneous problems:

8. Several positive integers are written on a chalk board. One can choose two of them, erase them, and replace them with their greatest common divisor and least common multiple. Prove that eventually the numbers on the board do not change.

9. Eight people sit around a lunch table. As it happens, each person's age is the average of the two persons' ages on his/her right and left. Show that their ages are equal.

10. (Putnam) Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$ and thereafter by the condition that

$$\det \begin{bmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{bmatrix} = n!$$

   for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

11. (Putnam) Let Q be an $n \times n$ real orthogonal matrix, and $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$ where I is the $n \times n$ identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.