Putnam practice session 5
Washington University in St. Louis — October 23, 2020

Limits and Series

Room 1: 1-4; Room 2: 5-7, Room 3: 8-10

1. (Putnam) Suppose that a sequence of positive numbers $a_1, a_2, a_3 \dots$ satisfies $a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Show that the series $\sum_{n=1}^{\infty} a_n$ diverges.

2. Calculate $\sum_{n=1}^{N} \frac{n}{(n+1)!}$.

3. Evaluate the following: $\sum_{n=1}^{\infty} \frac{2n+3}{n(n+1)^2(n+2)^2(n+3)}$

   Hint (highlight and/or copy+paste to make visible): ______________________

4. (Putnam) For given positive integer n, let $\langle n \rangle$ denote the integer closest to $\sqrt{n}$. Evaluate:
$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$

5. Show that $\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{1}{k}$ exists and determine its value.

6. (Putnam) Let r, s, t be non-negative integers with $r + s \leq t$. Prove that
$$\sum_{k=0}^{s} \frac{\binom{s}{k}}{\binom{t}{r+k}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}$$
   where $\binom{a}{b}$ is a binomial coefficient.

7. For real numbers a, b, prove that the sequence defined by the recurrence $x_1 > 0$, $x_{n+1} = \sqrt{a + bx_n}$ for $n \geq 1$, converges and determine its limit.

8. Calculate $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{1+3n}$.

9. For real x determine $\prod_{n=1}^{\infty} \dfrac{1 + 2\cos\frac{2x}{3^n}}{3}$.

10. Evaluate
$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2n + n^2m + 2mn}.$$