13 Review

CONCEPT CHECK

1. A vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. To find the derivative or integral, we can differentiate or integrate each component of the vector function.

2. The tip of the moving vector $\mathbf{r}(t)$ of a continuous vector function traces out a space curve.

3. The tangent vector to a smooth curve at a point P with position vector $\mathbf{r}(t)$ is the vector $\mathbf{r}'(t)$. The tangent line at P is the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

4. (a) (a) – (f) See Theorem 13.2.3.

5. Use Formula 13.3.2, or equivalently, 13.3.3.

6. (a) The curvature of a curve is $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector.

   (b) $\kappa(t) = \left|\dfrac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$ (c) $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ (d) $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$

7. (a) The unit normal vector: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. The binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

   (b) See the discussion preceding Example 7 in Section 13.3.

8. (a) If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

   (b) $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ where $a_T = v'$ and $a_N = \kappa v^2$.

9. See the statement of Kepler's Laws on page 892 [ET 868].

TRUE-FALSE QUIZ

1. True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u\,\mathbf{i} + 2u\,\mathbf{j} + 3u\,\mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$.

2. True. Parametric equations for the curve are $x = 0$, $y = t^2$, $z = 4t$, and since $t = z/4$ we have $y = t^2 = (z/4)^2$ or $y = \frac{1}{16}z^2$, $x = 0$. This is an equation of a parabola in the yz-plane.

3. False. The vector function represents a line, but the line does not pass through the origin; the x-component is 0 only for $t = 0$ which corresponds to the point $(0, 3, 0)$ not $(0, 0, 0)$.

4. True. See Theorem 13.2.2.

5. False. By Formula 5 of Theorem 13.2.3, $\dfrac{d}{dt}\,[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

6. False. For example, let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $|\mathbf{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \quad \Rightarrow \quad \dfrac{d}{dt}\,|\mathbf{r}(t)| = 0$, but

$|\mathbf{r}'(t)| = |\langle -\sin t, \cos t\rangle| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.

7. False. κ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length s, not with respect to t.

8. False. The binormal vector, by the definition given in Section 13.3, is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = -[\mathbf{N}(t) \times \mathbf{T}(t)]$.

9. True. At an inflection point where f is twice continuously differentiable we must have $f''(x) = 0$, and by Equation 13.3.11, the curvature is 0 there.

10. True. From Equation 13.3.9, $\kappa(t) = 0 \quad \Leftrightarrow \quad |\mathbf{T}'(t)| = 0 \quad \Leftrightarrow \quad \mathbf{T}'(t) = \mathbf{0}$ for all t. But then $\mathbf{T}(t) = \mathbf{C}$, a constant vector, which is true only for a straight line.

11. False. If $\mathbf{r}(t)$ is the position of a moving particle at time t and $|\mathbf{r}(t)| = 1$ then the particle lies on the unit circle or the unit sphere, but this does not mean that the speed $|\mathbf{r}'(t)|$ must be constant. As a counterexample, let $\mathbf{r}(t) = \langle t, \sqrt{1-t^2}\rangle$, then $\mathbf{r}'(t) = \langle 1, -t/\sqrt{1-t^2}\rangle$ and $|\mathbf{r}(t)| = \sqrt{t^2 + 1 - t^2} = 1$ but $|\mathbf{r}'(t)| = \sqrt{1 + t^2/(1-t^2)} = 1/\sqrt{1-t^2}$ which is not constant.

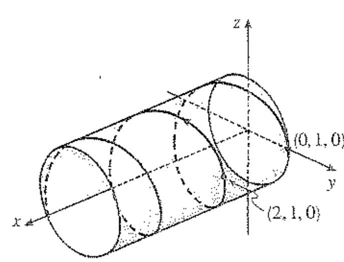
12. True. See Example 4 in Section 13.2.

13. True. See the discussion preceding Example 7 in Section 13.3.

14. False. For example, $\mathbf{r}_1(t) = \langle t, t\rangle$ and $\mathbf{r}_2(t) = \langle 2t, 2t\rangle$ both represent the same plane curve (the line $y = x$), but the tangent vector $\mathbf{r}_1'(t) = \langle 1, 1\rangle$ for all t, while $\mathbf{r}_2'(t) = \langle 2, 2\rangle$. In fact, different parametrizations give parallel tangent vectors at a point, but their magnitudes may differ.

EXERCISES

1. (a) The corresponding parametric equations for the curve are $x = t$, $y = \cos \pi t$, $z = \sin \pi t$. Since $y^2 + z^2 = 1$, the curve is contained in a circular cylinder with axis the x-axis. Since $x = t$, the curve is a helix.



(b) $\mathbf{r}(t) = t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\,\mathbf{j} + \pi \cos \pi t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}''(t) = -\pi^2 \cos \pi t\,\mathbf{j} - \pi^2 \sin \pi t\,\mathbf{k}$

2. (a) The expressions $\sqrt{2-t}$, $(e^t - 1)/t$, and $\ln(t+1)$ are all defined when $2 - t \geq 0 \quad \Rightarrow \quad t \leq 2, t \neq 0$, and $t + 1 > 0 \quad \Rightarrow \quad t > -1$. Thus the domain of $\mathbf{r}$ is $(-1, 0) \cup (0, 2]$.

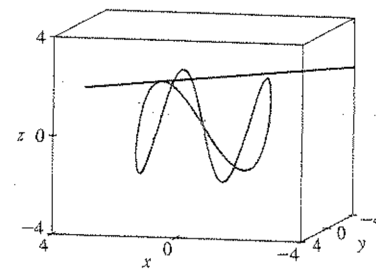
(b) $$\lim_{t\to 0} \mathbf{r}(t) = \left\langle \lim_{t\to 0} \sqrt{2-t}, \lim_{t\to 0} \frac{e^t - 1}{t}, \lim_{t\to 0} \ln(t+1)\right\rangle = \left\langle \sqrt{2-0}, \lim_{t\to 0} \frac{e^t}{1}, \ln(0+1)\right\rangle$$

$$= \langle \sqrt{2}, 1, 0\rangle \qquad \text{[using l'Hospital's Rule in the } y\text{-component]}$$

(c) $\mathbf{r}'(t) = \left\langle \dfrac{d}{dt}\sqrt{2-t}, \dfrac{d}{dt}\dfrac{e^t-1}{t}, \dfrac{d}{dt}\ln(t+1) \right\rangle = \left\langle -\dfrac{1}{2\sqrt{2-t}}, \dfrac{te^t-e^t+1}{t^2}, \dfrac{1}{t+1} \right\rangle$

3. The projection of the curve C of intersection onto the xy-plane is the circle $x^2+y^2=16$, $z=0$. So we can write $x=4\cos t,\ y=4\sin t,\ 0\le t\le 2\pi$. From the equation of the plane, we have $z=5-x=5-4\cos t$, so parametric equations for C are $x=4\cos t,\ y=4\sin t,\ z=5-4\cos t$, $0\le t\le 2\pi$, and the corresponding vector function is $\mathbf{r}(t)=4\cos t\,\mathbf{i}+4\sin t\,\mathbf{j}+(5-4\cos t)\,\mathbf{k}$, $0\le t\le 2\pi$.

4. The curve is given by $\mathbf{r}(t)=\langle 2\sin t, 2\sin 2t, 2\sin 3t\rangle$, so $\mathbf{r}'(t)=\langle 2\cos t, 4\cos 2t, 6\cos 3t\rangle$. The point $(1,\sqrt{3},2)$ corresponds to $t=\frac{\pi}{6}$ (or $\frac{\pi}{6}+2k\pi$, k an integer), so the tangent vector there is $\mathbf{r}'(\frac{\pi}{6})=\langle\sqrt{3},2,0\rangle$. Then the tangent line has direction vector $\langle\sqrt{3},2,0\rangle$ and includes the point $(1,\sqrt{3},2)$, so parametric equations are $x=1+\sqrt{3}\,t$, $y=\sqrt{3}+2t$, $z=2$.



5. $\int_0^1 (t^2\,\mathbf{i}+t\cos\pi t\,\mathbf{j}+\sin\pi t\,\mathbf{k})\,dt = \left(\int_0^1 t^2\,dt\right)\mathbf{i}+\left(\int_0^1 t\cos\pi t\,dt\right)\mathbf{j}+\left(\int_0^1 \sin\pi t\,dt\right)\mathbf{k}$

$$= \left[\tfrac{1}{3}t^3\right]_0^1\mathbf{i}+\left(\left.\tfrac{t}{\pi}\sin\pi t\right]_0^1-\int_0^1\tfrac{1}{\pi}\sin\pi t\,dt\right)\mathbf{j}+\left[-\tfrac{1}{\pi}\cos\pi t\right]_0^1\mathbf{k}$$

$$= \tfrac{1}{3}\mathbf{i}+\left[\tfrac{1}{\pi^2}\cos\pi t\right]_0^1\mathbf{j}+\tfrac{2}{\pi}\mathbf{k}=\tfrac{1}{3}\mathbf{i}-\tfrac{2}{\pi^2}\mathbf{j}+\tfrac{2}{\pi}\mathbf{k}$$

where we integrated by parts in the y-component.

6. (a) C intersects the xz-plane where $y=0 \;\Rightarrow\; 2t-1=0 \;\Rightarrow\; t=\frac{1}{2}$, so the point is $\left(2-\left(\frac{1}{2}\right)^3, 0, \ln\frac{1}{2}\right)=\left(\frac{15}{8},0,-\ln 2\right)$.

(b) The curve is given by $\mathbf{r}(t)=\langle 2-t^3, 2t-1, \ln t\rangle$, so $\mathbf{r}'(t)=\langle -3t^2, 2, 1/t\rangle$. The point $(1,1,0)$ corresponds to $t=1$, so the tangent vector there is $\mathbf{r}'(1)=\langle -3,2,1\rangle$. Then the tangent line has direction vector $\langle -3,2,1\rangle$ and includes the point $(1,1,0)$, so parametric equations are $x=1-3t$, $y=1+2t$, $z=t$.

(c) The normal plane has normal vector $\mathbf{r}'(1)=\langle -3,2,1\rangle$ and equation $-3(x-1)+2(y-1)+z=0$ or $3x-2y-z=1$.

7. $\mathbf{r}(t)=\langle t^2,t^3,t^4\rangle \;\Rightarrow\; \mathbf{r}'(t)=\langle 2t,3t^2,4t^3\rangle \;\Rightarrow\; |\mathbf{r}'(t)|=\sqrt{4t^2+9t^4+16t^6}$ and $L=\int_0^3|\mathbf{r}'(t)|\,dt=\int_0^3\sqrt{4t^2+9t^4+16t^6}\,dt$. Using Simpson's Rule with $f(t)=\sqrt{4t^2+9t^4+16t^6}$ and $n=6$ we have $\Delta t=\frac{3-0}{6}=\frac{1}{2}$ and

$$L\approx\tfrac{\Delta t}{3}\left[f(0)+4f(\tfrac{1}{2})+2f(1)+4f(\tfrac{3}{2})+2f(2)+4f(\tfrac{5}{2})+f(3)\right]$$

$$=\tfrac{1}{6}\Big[\sqrt{0+0+0}+4\cdot\sqrt{4(\tfrac{1}{2})^2+9(\tfrac{1}{2})^4+16(\tfrac{1}{2})^6}+2\cdot\sqrt{4(1)^2+9(1)^4+16(1)^6}$$

$$+4\cdot\sqrt{4(\tfrac{3}{2})^2+9(\tfrac{3}{2})^4+16(\tfrac{3}{2})^6}+2\cdot\sqrt{4(2)^2+9(2)^4+16(2)^6}$$

$$+4\cdot\sqrt{4(\tfrac{5}{2})^2+9(\tfrac{5}{2})^4+16(\tfrac{5}{2})^6}+\sqrt{4(3)^2+9(3)^4+16(3)^6}\Big]$$

$$\approx 86.631$$

8. $\mathbf{r}'(t) = \left\langle 3t^{1/2}, -2\sin 2t, 2\cos 2t \right\rangle$, $|\mathbf{r}'(t)| = \sqrt{9t + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{9t+4}$.

Thus $L = \int_0^1 \sqrt{9t+4}\,dt = \int_4^{13} \frac{1}{9}u^{1/2}\,du = \frac{1}{9} \cdot \frac{2}{3}u^{3/2}\Big]_4^{13} = \frac{2}{27}(13^{3/2} - 8)$.

9. The angle of intersection of the two curves, θ, is the angle between their respective tangents at the point of intersection.

For both curves the point $(1, 0, 0)$ occurs when $t = 0$.

$\mathbf{r}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \;\Rightarrow\; \mathbf{r}_1'(0) = \mathbf{j} + \mathbf{k}$ and $\mathbf{r}_2'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}_2'(0) = \mathbf{i}$.

$\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(0) = (\mathbf{j} + \mathbf{k}) \cdot \mathbf{i} = 0$. Therefore, the curves intersect in a right angle, that is, $\theta = \frac{\pi}{2}$.

10. The parametric value corresponding to the point $(1, 0, 1)$ is $t = 0$.

$\mathbf{r}'(t) = e^t\,\mathbf{i} + e^t(\cos t + \sin t)\,\mathbf{j} + e^t(\cos t - \sin t)\,\mathbf{k} \;\Rightarrow\; |\mathbf{r}'(t)| = e^t\sqrt{1 + (\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \sqrt{3}\,e^t$

and $s(t) = \int_0^t e^u\sqrt{3}\,du = \sqrt{3}(e^t - 1) \;\Rightarrow\; t = \ln\left(1 + \frac{1}{\sqrt{3}}s\right)$.

Therefore, $\mathbf{r}(t(s)) = \left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{i} + \left(1 + \frac{1}{\sqrt{3}}s\right)\sin\ln\left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{j} + \left(1 + \frac{1}{\sqrt{3}}s\right)\cos\ln\left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{k}$.

11. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle t^2, t, 1\rangle}{|\langle t^2, t, 1\rangle|} = \dfrac{\langle t^2, t, 1\rangle}{\sqrt{t^4 + t^2 + 1}}$

(b) $$\begin{aligned}\mathbf{T}'(t) &= -\tfrac{1}{2}(t^4 + t^2 + 1)^{-3/2}(4t^3 + 2t)\langle t^2, t, 1\rangle + (t^4 + t^2 + 1)^{-1/2}\langle 2t, 1, 0\rangle \\ &= \frac{-2t^3 - t}{(t^4 + t^2 + 1)^{3/2}}\langle t^2, t, 1\rangle + \frac{1}{(t^4 + t^2 + 1)^{1/2}}\langle 2t, 1, 0\rangle \\ &= \frac{\langle -2t^5 - t^3, -2t^4 - t^2, -2t^3 - t\rangle + \langle 2t^5 + 2t^3 + 2t, t^4 + t^2 + 1, 0\rangle}{(t^4 + t^2 + 1)^{3/2}} = \frac{\langle t^3 + 2t, -t^4 + 1, -2t^3 - t\rangle}{(t^4 + t^2 + 1)^{3/2}}\end{aligned}$$

$$|\mathbf{T}'(t)| = \frac{\sqrt{t^6 + 4t^4 + 4t^2 + t^8 - 2t^4 + 1 + 4t^6 + 4t^4 + t^2}}{(t^4 + t^2 + 1)^{3/2}} = \frac{\sqrt{t^8 + 5t^6 + 6t^4 + 5t^2 + 1}}{(t^4 + t^2 + 1)^{3/2}}$$ and

$$\mathbf{N}(t) = \frac{\langle t^3 + 2t, 1 - t^4, -2t^3 - t\rangle}{\sqrt{t^8 + 5t^6 + 6t^4 + 5t^2 + 1}}.$$

(c) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{t^8 + 5t^6 + 6t^4 + 5t^2 + 1}}{(t^4 + t^2 + 1)^2}$ or $\dfrac{\sqrt{t^4 + 4t^2 + 1}}{(t^4 + t^2 + 1)^{3/2}}$

12. Using Exercise 13.3.42, we have $\mathbf{r}'(t) = \langle -3\sin t, 4\cos t\rangle$, $\mathbf{r}''(t) = \langle -3\cos t, -4\sin t\rangle$,

$|\mathbf{r}'(t)|^3 = \left(\sqrt{9\sin^2 t + 4\cos^2 t}\right)^3$ and then

$$\kappa(t) = \frac{|(-3\sin t)(-4\sin t) - (4\cos t)(-3\cos t)|}{(9\sin^2 t + 16\cos^2 t)^{3/2}} = \frac{12}{(9\sin^2 t + 16\cos^2 t)^{3/2}}.$$

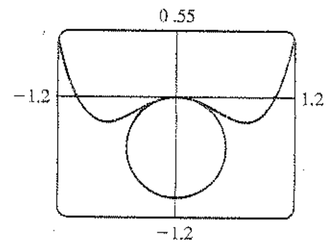
At $(3, 0)$, $t = 0$ and $\kappa(0) = 12/(16)^{3/2} = \frac{12}{64} = \frac{3}{16}$. At $(0, 4)$, $t = \frac{\pi}{2}$ and $\kappa\left(\frac{\pi}{2}\right) = 12/9^{3/2} = \frac{12}{27} = \frac{4}{9}$.

13. $y' = 4x^3$, $y'' = 12x^2$ and $\kappa(x) = \dfrac{|y''|}{[1 + (y')^2]^{3/2}} = \dfrac{|12x^2|}{(1 + 16x^6)^{3/2}}$, so $\kappa(1) = \dfrac{12}{17^{3/2}}$.

14. $\kappa(x) = \dfrac{|12x^2 - 2|}{[1 + (4x^3 - 2x)^2]^{3/2}} \quad \Rightarrow \quad \kappa(0) = 2.$

So the osculating circle has radius $\frac{1}{2}$ and center $\left(0, -\frac{1}{2}\right)$.

Thus its equation is $x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$.



15. $\mathbf{r}(t) = \langle \sin 2t, t, \cos 2t \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 2\cos 2t, 1, -2\sin 2t \rangle \quad \Rightarrow \quad \mathbf{T}(t) = \frac{1}{\sqrt{5}} \langle 2\cos 2t, 1, -2\sin 2t \rangle \quad \Rightarrow$

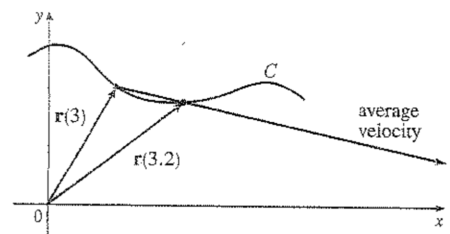
$\mathbf{T}'(t) = \frac{1}{\sqrt{5}} \langle -4\sin 2t, 0, -4\cos 2t \rangle \quad \Rightarrow \quad \mathbf{N}(t) = \langle -\sin 2t, 0, -\cos 2t \rangle$. So $\mathbf{N} = \mathbf{N}(\pi) = \langle 0, 0, -1 \rangle$ and

$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}} \langle -1, 2, 0 \rangle$. So a normal to the osculating plane is $\langle -1, 2, 0 \rangle$ and an equation is

$-1(x - 0) + 2(y - \pi) + 0(z - 1) = 0$ or $x - 2y + 2\pi = 0$.

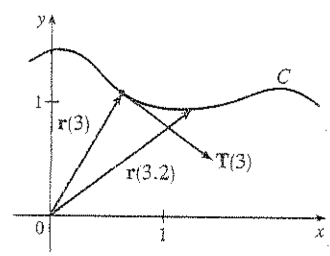
16. (a) The average velocity over $[3, 3.2]$ is given by

$\dfrac{\mathbf{r}(3.2) - \mathbf{r}(3)}{3.2 - 3} = 5[\mathbf{r}(3.2) - \mathbf{r}(3)]$, so we draw a vector with the same direction but 5 times the length of the vector $[\mathbf{r}(3.2) - \mathbf{r}(3)]$.



(b) $\mathbf{v}(3) = \mathbf{r}'(3) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}(3 + h) - \mathbf{r}(3)}{h}$

(c) $\mathbf{T}(3) = \dfrac{\mathbf{r}'(3)}{|\mathbf{r}'(3)|}$, a unit vector in the same direction as $\mathbf{r}'(3)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(3)$, pointing in the direction corresponding to increasing t, and with length 1.



17. $\mathbf{r}(t) = t\ln t\,\mathbf{i} + t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad \mathbf{v}(t) = \mathbf{r}'(t) = (1 + \ln t)\,\mathbf{i} + \mathbf{j} - e^{-t}\,\mathbf{k},$

$|\mathbf{v}(t)| = \sqrt{(1 + \ln t)^2 + 1^2 + (-e^{-t})^2} = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}, \quad \mathbf{a}(t) = \mathbf{v}'(t) = \frac{1}{t}\,\mathbf{i} + e^{-t}\,\mathbf{k}$

18. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (6t\,\mathbf{i} + 12t^2\,\mathbf{j} - 6t\,\mathbf{k})\,dt = 3t^2\,\mathbf{i} + 4t^3\,\mathbf{j} - 3t^2\,\mathbf{k} + \mathbf{C}$, but $\mathbf{i} - \mathbf{j} + 3\,\mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{C}$,

so $\mathbf{C} = \mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v}(t) = (3t^2 + 1)\,\mathbf{i} + (4t^3 - 1)\,\mathbf{j} + (3 - 3t^2)\,\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k} + \mathbf{D}$.

But $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{D} = \mathbf{0}$ and $\mathbf{r}(t) = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k}$.

19. We set up the axes so that the shot leaves the athlete's hand 7 ft above the origin. Then we are given $\mathbf{r}(0) = 7\mathbf{j}$, $|\mathbf{v}(0)| = 43$ ft/s, and $\mathbf{v}(0)$ has direction given by a $45°$ angle of elevation. Then a unit vector in the direction of $\mathbf{v}(0)$ is $\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \quad \Rightarrow \quad \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Assuming air resistance is negligible, the only external force is due to gravity, so as in Example 13.4.5 we have $\mathbf{a} = -g\,\mathbf{j}$ where here $g \approx 32$ ft/s^2. Since $\mathbf{v}'(t) = \mathbf{a}(t)$, we integrate, giving $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}$

where $\mathbf{C} = \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \Rightarrow \mathbf{v}(t) = \frac{43}{\sqrt{2}}\,\mathbf{i} + \left(\frac{43}{\sqrt{2}} - gt\right)\mathbf{j}$. Since $\mathbf{r}'(t) = \mathbf{v}(t)$ we integrate again, so

$\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{D}$. But $\mathbf{D} = \mathbf{r}(0) = 7\,\mathbf{j} \Rightarrow \mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7\right)\mathbf{j}$.

(a) At 2 seconds, the shot is at $\mathbf{r}(2) = \frac{43}{\sqrt{2}}(2)\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}(2) - \frac{1}{2}g(2)^2 + 7\right)\mathbf{j} \approx 60.8\,\mathbf{i} + 3.8\,\mathbf{j}$, so the shot is about 3.8 ft above the ground, at a horizontal distance of 60.8 ft from the athlete.

(b) The shot reaches its maximum height when the vertical component of velocity is 0: $\frac{43}{\sqrt{2}} - gt = 0 \Rightarrow$

$t = \dfrac{43}{\sqrt{2}\,g} \approx 0.95$ s. Then $\mathbf{r}(0.95) \approx 28.9\,\mathbf{i} + 21.4\,\mathbf{j}$, so the maximum height is approximately 21.4 ft.

(c) The shot hits the ground when the vertical component of $\mathbf{r}(t)$ is 0, so $\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7 = 0 \Rightarrow$

$-16t^2 + \frac{43}{\sqrt{2}}t + 7 = 0 \Rightarrow t \approx 2.11$ s. $\mathbf{r}(2.11) \approx 64.2\,\mathbf{i} - 0.08\,\mathbf{j}$, thus the shot lands approximately 64.2 ft from the athlete.

20. $\mathbf{r}'(t) = \mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{r}''(t) = 2\,\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1 + 4 + 4t^2} = \sqrt{4t^2 + 5}$.

Then $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t}{\sqrt{4t^2 + 5}}$ and $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{|4\,\mathbf{i} - 2\,\mathbf{j}|}{\sqrt{4t^2 + 5}} = \dfrac{2\sqrt{5}}{\sqrt{4t^2 + 5}}$.

21. (a) Instead of proceeding directly, we use Formula 3 of Theorem 13.2.3: $\mathbf{r}(t) = t\,\mathbf{R}(t) \Rightarrow$

$\mathbf{v} = \mathbf{r}'(t) = \mathbf{R}(t) + t\,\mathbf{R}'(t) = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j} + t\,\mathbf{v}_d$.

(b) Using the same method as in part (a) and starting with $\mathbf{v} = \mathbf{R}(t) + t\,\mathbf{R}'(t)$, we have

$\mathbf{a} = \mathbf{v}' = \mathbf{R}'(t) + \mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{v}_d + t\,\mathbf{a}_d$.

(c) Here we have $\mathbf{r}(t) = e^{-t}\cos\omega t\,\mathbf{i} + e^{-t}\sin\omega t\,\mathbf{j} = e^{-t}\,\mathbf{R}(t)$. So, as in parts (a) and (b),

$\mathbf{v} = \mathbf{r}'(t) = e^{-t}\,\mathbf{R}'(t) - e^{-t}\,\mathbf{R}(t) = e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] \Rightarrow$

$$\begin{aligned}\mathbf{a} = \mathbf{v}' &= e^{-t}[\mathbf{R}''(t) - \mathbf{R}'(t)] - e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] = e^{-t}[\mathbf{R}''(t) - 2\,\mathbf{R}'(t) + \mathbf{R}(t)] \\ &= e^{-t}\,\mathbf{a}_d - 2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}\end{aligned}$$

Thus, the Coriolis acceleration (the sum of the "extra" terms not involving $\mathbf{a}_d$) is $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$.

22. (a) $F(x) = \begin{cases} 1 & \text{if } x \le 0 \\ \sqrt{1 - x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ \sqrt{2} - x & \text{if } x \ge \frac{1}{\sqrt{2}} \end{cases} \Rightarrow F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ -x/\sqrt{1 - x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -1 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases} \Rightarrow$

$$F''(x) = \begin{cases} 0 & \text{if } x < 0 \\ -1/(1 - x^2)^{3/2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ 0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

since $\dfrac{d}{dx}[-x(1 - x^2)^{-1/2}] = -(1 - x^2)^{-1/2} - x^2(1 - x^2)^{-3/2} = -(1 - x^2)^{-3/2}$.

Now $\lim\limits_{x \to 0^+} \sqrt{1 - x^2} = 1 = F(0)$ and $\lim\limits_{x \to (1/\sqrt{2})^-} \sqrt{1 - x^2} = \frac{1}{\sqrt{2}} = F\left(\frac{1}{\sqrt{2}}\right)$, so F is continuous. Also, since