8. (a) $m = \iiint_E \rho(x, y, z)\,dV$

(b) $M_{yz} = \iiint_E x\rho(x, y, z)\,dV$, $M_{xz} = \iiint_E y\rho(x, y, z)\,dV$, $M_{xy} = \iiint_E z\rho(x, y, z)\,dV$.

(c) The center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \dfrac{M_{yz}}{m}$, $\overline{y} = \dfrac{M_{xz}}{m}$, and $\overline{z} = \dfrac{M_{xy}}{m}$.

(d) $I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\,dV$, $I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\,dV$, $I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\,dV$.

9. (a) See Formula 15.8.4 and the accompanying discussion.

(b) See Formula 15.9.3 and the accompanying discussion.

(c) We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates or if the triple integral is easier to evaluate using cylindrical or spherical coordinates.

10. (a) $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

(b) See (9) and the accompanying discussion in Section 15.10.

(c) See (13) and the accompanying discussion in Section 15.10.

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

2. False. $\int_0^1 \int_0^x \sqrt{x + y^2}\,dy\,dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1 \int_y^1 \sqrt{x + y^2}\,dx\,dy$.

3. True by Equation 15.2.5.

4. $\int_{-1}^1 \int_0^1 e^{x^2+y^2} \sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^1 e^{y^2} \sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2} \sin y$ is an odd function.

Therefore the statement is true.

5. True. By Equation 15.2.5 we can write $\int_0^1 \int_0^1 f(x)\,f(y)\,dy\,dx = \int_0^1 f(x)\,dx \int_0^1 f(y)\,dy$. But $\int_0^1 f(y)\,dy = \int_0^1 f(x)\,dx$ so this becomes $\int_0^1 f(x)\,dx \int_0^1 f(x)\,dx = \left[\int_0^1 f(x)\,dx\right]^2$.

6. This statement is true because in the given region, $(x^2 + \sqrt{y}) \sin(x^2y^2) \le (1 + 2)(1) = 3$, so

$\int_1^4 \int_0^1 (x^2 + \sqrt{y}) \sin(x^2y^2)\,dx\,dy \le \int_1^4 \int_0^1 3\,dA = 3A(D) = 3(3) = 9$.

7. True: $\iint_D \sqrt{4 - x^2 - y^2}\,dA$ = the volume under the surface $x^2 + y^2 + z^2 = 4$ and above the xy-plane

$= \frac{1}{2}$ (the volume of the sphere $x^2 + y^2 + z^2 = 4$) $= \frac{1}{2} \cdot \frac{4}{3}\pi(2)^3 = \frac{16}{3}\pi$

8. True. The moment of inertia about the z-axis of a solid E with constant density k is

$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iiint_E (kr^2)\,r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta.$

9. The volume enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$ is, in cylindrical coordinates,

$V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta \neq \int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$, so the assertion is false.

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m = n = 3$ and the sample points the upper right corners of each square, so

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A$$
$$= \Delta A\,[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)+f(2,3)+f(3,1)+f(3,2)+f(3,3)]$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

2. As in Exercise 1, we have $m = n = 3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(\overline{x}_i,\overline{y}_j)\,\Delta A$$
$$= \Delta A\,[f(0.5,0.5)+(0.5,1.5)+(0.5,2.5)+(1.5,0.5)+f(1.5,1.5)$$
$$+ f(1.5,2.5)+(2.5,0.5)+f(2.5,1.5)+f(2.5,2.5)]$$
$$\approx 1[1.2+2.5+5.0+3.2+4.5+7.1+5.2+6.5+9.0] = 44.2$$

3. $\int_1^2\int_0^2 (y+2xe^y)\,dx\,dy = \int_1^2 \left[xy+x^2e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y+4e^y)\,dy = \left[y^2+4e^y\right]_1^2$
$= 4+4e^2-1-4e = 4e^2-4e+3$

4. $\int_0^1\int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 (e^y-1)\,dy = \left[e^y-y\right]_0^1 = e-2$

5. $\int_0^1\int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big]_0^1 = \frac{1}{2}\sin 1$

6. $\int_0^1\int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x}-x^4)\,dx = \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ [integrate by parts in the first term]
$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

7. $\int_0^\pi\int_0^1\int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi\int_0^1 \left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi\int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$
$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

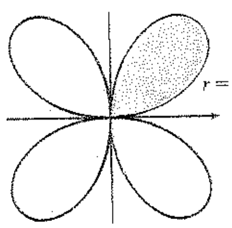
8. $\int_0^1 \int_0^y \int_x^1 6xyz\,dz\,dx\,dy = \int_0^1 \int_0^y \left[3xyz^2\right]_{z=x}^{z=1} dx\,dy = \int_0^1 \int_0^y (3xy - 3x^3y)\,dx\,dy$

$= \int_0^1 \left[\frac{3}{2}x^2y - \frac{3}{4}x^4y\right]_{x=0}^{x=y} dy = \int_0^1 \left(\frac{3}{2}y^3 - \frac{3}{4}y^5\right) dy = \left[\frac{3}{8}y^4 - \frac{1}{8}y^6\right]_0^1 = \frac{1}{4}$

9. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 2 \le r \le 4, 0 \le \theta \le \pi\}$. Thus

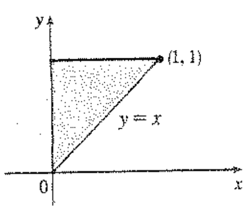
$\iint_R f(x, y)\,dA = \int_0^\pi \int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x, y) \mid y - 4 \le x \le 4 - y, 0 \le y \le 4\}$ and $\iint_R f(x, y)\,dA = \int_0^4 \int_{y-4}^{4-y} f(x, y)\,dx\,dy$.

11. 

The region whose area is given by $\int_0^{\pi/2} \int_0^{\sin 2\theta} r\,dr\,d\theta$ is $\{(r, \theta) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le \sin 2\theta\}$, which is the region contained in the loop in the first quadrant of the four-leaved rose $r = \sin 2\theta$.

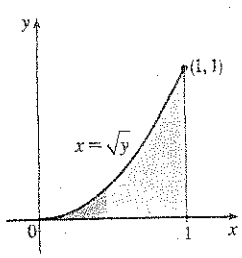
12. The solid is $\{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\}$ which is the region in the first octant on or between the two spheres $\rho = 1$ and $\rho = 2$.

13. 

$\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx = \int_0^1 \int_0^y \cos(y^2)\,dx\,dy$

$= \int_0^1 \cos(y^2)\left[x\right]_{x=0}^{x=y} dy = \int_0^1 y\cos(y^2)\,dy$

$= \left[\frac{1}{2}\sin(y^2)\right]_0^1 = \frac{1}{2}\sin 1$

14. 

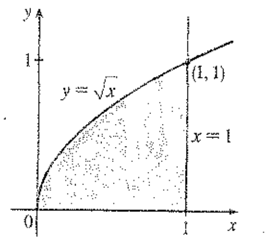
$$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy = \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3}\,dy\,dx = \int_0^1 \frac{e^{x^2}}{x^3}\left[\frac{1}{2}y^2\right]_{y=0}^{y=x^2} dx$$

$$= \int_0^1 \frac{1}{2}xe^{x^2}\,dx = \frac{1}{4}e^{x^2}\Big]_0^1 = \frac{1}{4}(e - 1)$$

15. $\iint_R ye^{xy}\,dA = \int_0^3 \int_0^2 ye^{xy}\,dx\,dy = \int_0^3 \left[e^{xy}\right]_{x=0}^{x=2} dy = \int_0^3 (e^{2y} - 1)\,dy = \left[\frac{1}{2}e^{2y} - y\right]_0^3 = \frac{1}{2}e^6 - 3 - \frac{1}{2} = \frac{1}{2}e^6 - \frac{7}{2}$

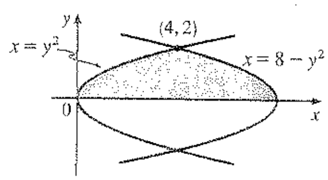
16. $\iint_D xy\,dA = \int_0^1 \int_{y^2}^{y+2} xy\,dx\,dy = \int_0^1 y\left[\frac{1}{2}x^2\right]_{x=y^2}^{x=y+2} dy = \frac{1}{2}\int_0^1 y((y+2)^2 - y^4)\,dy$

$= \frac{1}{2}\int_0^1 (y^3 + 4y^2 + 4y - y^5)\,dy = \frac{1}{2}\left[\frac{1}{4}y^4 + \frac{4}{3}y^3 + 2y^2 - \frac{1}{6}y^6\right]_0^1 = \frac{41}{24}$

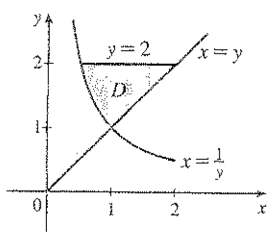
17. 

$$\iint_D \frac{y}{1+x^2}\,dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}\,dx$$
$$= \tfrac{1}{2}\int_0^1 \frac{x}{1+x^2}\,dx = \left[\tfrac{1}{4}\ln(1+x^2)\right]_0^1 = \tfrac{1}{4}\ln 2$$

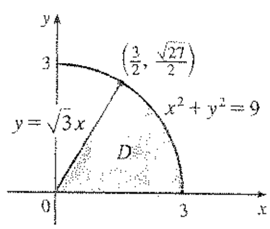
18. $$\iint_D \frac{1}{1+x^2}\,dA = \int_0^1 \int_x^1 \frac{1}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\left[y\right]_{y=x}^{y=1}\,dx = \int_0^1 \frac{1-x}{1+x^2}\,dx = \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right)dx$$
$$= \left[\tan^{-1}x - \tfrac{1}{2}\ln(1+x^2)\right]_0^1 = \tan^{-1}1 - \tfrac{1}{2}\ln 2 - \left(\tan^{-1}0 - \tfrac{1}{2}\ln 1\right) = \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$$

19. 

$$\iint_D y\,dA = \int_0^2 \int_{y^2}^{8-y^2} y\,dx\,dy$$
$$= \int_0^2 y\left[x\right]_{x=y^2}^{x=8-y^2}\,dy = \int_0^2 y(8-y^2-y^2)\,dy$$
$$= \int_0^2 (8y - 2y^3)\,dy = \left[4y^2 - \tfrac{1}{2}y^4\right]_0^2 = 8$$

20. 

$$\iint_D y\,dA = \int_1^2 \int_{1/y}^{y} y\,dx\,dy = \int_1^2 y\left(y - \frac{1}{y}\right)dy$$
$$= \int_1^2 (y^2 - 1)\,dy = \left[\tfrac{1}{3}y^3 - y\right]_1^2$$
$$= \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right) = \tfrac{4}{3}$$

21. 

$$\iint_D (x^2+y^2)^{3/2}\,dA = \int_0^{\pi/3} \int_0^3 (r^2)^{3/2}\,r\,dr\,d\theta$$
$$= \int_0^{\pi/3} d\theta \int_0^3 r^4\,dr = \left[\theta\right]_0^{\pi/3}\left[\tfrac{1}{5}r^5\right]_0^3$$
$$= \frac{\pi}{3}\frac{3^5}{5} = \frac{81\pi}{5}$$

22. $$\iint_D x\,dA = \int_0^{\pi/2}\int_1^{\sqrt{2}} (r\cos\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\,d\theta \int_1^{\sqrt{2}} r^2\,dr = \left[\sin\theta\right]_0^{\pi/2}\left[\tfrac{1}{3}r^3\right]_1^{\sqrt{2}}$$
$$= 1\cdot\tfrac{1}{3}(2^{3/2}-1) = \tfrac{1}{3}(2^{3/2}-1)$$

23. $$\iiint_E xy\,dV = \int_0^3\int_0^x\int_0^{x+y} xy\,dz\,dy\,dx = \int_0^3\int_0^x xy\left[z\right]_{z=0}^{z=x+y}\,dy\,dx = \int_0^3\int_0^x xy(x+y)\,dy\,dx$$
$$= \int_0^3\int_0^x (x^2y + xy^2)\,dy\,dx = \int_0^3 \left[\tfrac{1}{2}x^2y^2 + \tfrac{1}{3}xy^3\right]_{y=0}^{y=x}\,dx = \int_0^3 \left(\tfrac{1}{2}x^4 + \tfrac{1}{3}x^4\right)dx$$
$$= \tfrac{5}{6}\int_0^3 x^4\,dx = \left[\tfrac{1}{6}x^5\right]_0^3 = \tfrac{81}{2} = 40.5$$

24. $\iiint_T xy\,dV = \int_0^{1/3}\int_0^{1-3x}\int_0^{1-3x-y} xy\,dz\,dy\,dx = \int_0^{1/3}\int_0^{1-3x} xy(1-3x-y)\,dy\,dx$

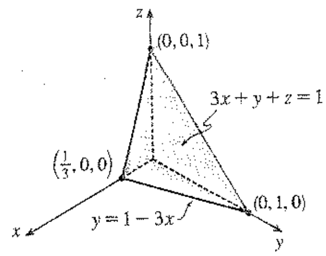
$= \int_0^{1/3}\int_0^{1-3x}(xy - 3x^2y - xy^2)\,dy\,dx$

$= \int_0^{1/3}\left[\frac{1}{2}xy^2 - \frac{3}{2}x^2y^2 - \frac{1}{3}xy^3\right]_{y=0}^{y=1-3x} dx$

$= \int_0^{1/3}\left[\frac{1}{2}x(1-3x)^2 - \frac{3}{2}x^2(1-3x)^2 - \frac{1}{3}x(1-3x)^3\right] dx$

$= \int_0^{1/3}\left(\frac{1}{6}x - \frac{3}{2}x^2 + \frac{9}{2}x^3 - \frac{9}{2}x^4\right) dx$

$= \frac{1}{12}x^2 - \frac{1}{2}x^3 + \frac{9}{8}x^4 - \frac{9}{10}x^5\Big]_0^{1/3} = \frac{1}{1080}$



25. $\iiint_E y^2z^2\,dV = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2(1-y^2-z^2)\,dz\,dy$

$= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(r^2\sin^2\theta)(1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{4}\sin^2 2\theta(r^5 - r^7)\,dr\,d\theta$

$= \int_0^{2\pi}\frac{1}{8}(1-\cos 4\theta)\left[\frac{1}{6}r^6 - \frac{1}{8}r^8\right]_{r=0}^{r=1} d\theta = \frac{1}{192}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{2\pi}{192} = \frac{\pi}{96}$

26. $\iiint_E z\,dV = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{2-y} z\,dx\,dz\,dy = \int_0^1\int_0^{\sqrt{1-y^2}}(2-y)z\,dz\,dy = \int_0^1 \frac{1}{2}(2-y)(1-y^2)\,dy$

$= \int_0^1 \frac{1}{2}(2 - y - 2y^2 + y^3)\,dy = \frac{13}{24}$

27. $\iiint_E yz\,dV = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^y yz\,dz\,dy\,dx = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}} \frac{1}{2}y^3\,dy\,dx = \int_0^{\pi}\int_0^2 \frac{1}{2}r^3(\sin^3\theta)\,r\,dr\,d\theta$

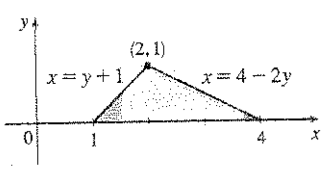
$= \frac{16}{5}\int_0^{\pi}\sin^3\theta\,d\theta = \frac{16}{5}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{64}{15}$

28. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho^3\cos^3\phi)\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi} d\theta\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi\int_0^1 \rho^6\,d\rho = 2\pi\left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}\left(\frac{1}{7}\right) = \frac{\pi}{14}$

29. $V = \int_0^2\int_1^4 (x^2 + 4y^2)\,dy\,dx = \int_0^2\left[x^2y + \frac{4}{3}y^3\right]_{y=1}^{y=4} dx = \int_0^2 (3x^2 + 84)\,dx = 176$

30.

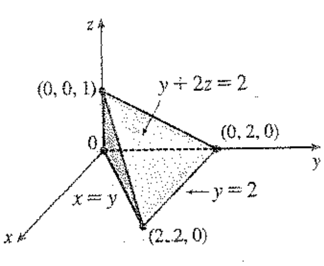


$V = \int_0^1\int_{y+1}^{4-2y}\int_0^{x^2y} dz\,dx\,dy = \int_0^1\int_{y+1}^{4-2y} x^2y\,dx\,dy$

$= \int_0^1 \frac{1}{3}\left[(4-2y)^3y - (y+1)^3y\right] dy$

$= \int_0^1 3(-y^4 + 5y^3 - 11y^2 + 7y)\,dy = 3\left(-\frac{1}{5} + \frac{5}{4} - \frac{11}{3} + \frac{7}{2}\right) = \frac{53}{20}$

31.



$V = \int_0^2\int_0^y\int_0^{(2-y)/2} dz\,dx\,dy = \int_0^2\int_0^y \left(1 - \frac{1}{2}y\right) dx\,dy$

$= \int_0^2 \left(y - \frac{1}{2}y^2\right) dy = \frac{2}{3}$

32. $V = \int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (3r - r^2\sin\theta)\,dr\,d\theta = \int_0^{2\pi}\left[6 - \frac{8}{3}\sin\theta\right] d\theta = 6\theta\Big]_0^{2\pi} + 0 = 12\pi$

33. Using the wedge above the plane $z = 0$ and below the plane $z = mx$ and noting that we have the same volume for $m < 0$ as for $m > 0$ (so use $m > 0$), we have

$$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}} mx\,dx\,dy = 2\int_0^{a/3} \tfrac{1}{2}m(a^2 - 9y^2)\,dy = m\left[a^2y - 3y^3\right]_0^{a/3} = m\left(\tfrac{1}{3}a^3 - \tfrac{1}{9}a^3\right) = \tfrac{2}{9}ma^3.$$

34. The paraboloid and the half-cone intersect when $x^2 + y^2 = \sqrt{x^2+y^2}$, that is when $x^2 + y^2 = 1$ or 0. So

$$V = \iint\limits_{x^2+y^2\le 1} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dA = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2 - r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{1}{3} - \tfrac{1}{4}\right)d\theta = \tfrac{1}{12}(2\pi) = \tfrac{\pi}{6}.$$

35. (a) $m = \int_0^1\int_0^{1-y^2} y\,dx\,dy = \int_0^1 (y - y^3)\,dy = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(b) $M_y = \int_0^1\int_0^{1-y^2} xy\,dx\,dy = \int_0^1 \frac{1}{2}y(1-y^2)^2\,dy = -\frac{1}{12}(1-y^2)^3\Big]_0^1 = \frac{1}{12}$,

$M_x = \int_0^1\int_0^{1-y^2} y^2\,dx\,dy = \int_0^1 (y^2 - y^4)\,dy = \frac{2}{15}$. Hence $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{8}{15}\right)$.

(c) $I_x = \int_0^1\int_0^{1-y^2} y^3\,dx\,dy = \int_0^1 (y^3 - y^5)\,dy = \frac{1}{12}$,

$I_y = \int_0^1\int_0^{1-y^2} yx^2\,dx\,dy = \int_0^1 \frac{1}{3}y(1-y^2)^3\,dy = -\frac{1}{24}(1-y^2)^4\Big]_0^1 = \frac{1}{24}$,

$I_0 = I_x + I_y = \frac{1}{8}$, $\overline{\overline{y}}^2 = \frac{1/12}{1/4} = \frac{1}{3} \;\Rightarrow\; \overline{\overline{y}} = \frac{1}{\sqrt{3}}$, and $\overline{\overline{x}}^2 = \frac{1/24}{1/4} = \frac{1}{6} \;\Rightarrow\; \overline{\overline{x}} = \frac{1}{\sqrt{6}}$.

36. (a) $m = \frac{1}{4}\pi K a^2$ where K is constant,

$M_y = \iint_{x^2+y^2\le a^2} Kx\,dA = K\int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{1}{3}Ka^3\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{3}a^3K$, and

$M_x = K\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{1}{3}a^3K$ [by symmetry $M_y = M_x$].

Hence the centroid is $(\overline{x}, \overline{y}) = \left(\frac{4}{3\pi}a, \frac{4}{3\pi}a\right)$.

(b) $m = \int_0^{\pi/2}\int_0^a r^4\cos\theta\sin^2\theta\,dr\,d\theta = \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}\left(\frac{1}{5}a^5\right) = \frac{1}{15}a^5$,

$M_y = \int_0^{\pi/2}\int_0^a r^5\cos^2\theta\sin^2\theta\,dr\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{96}\pi a^6$, and

$M_x = \int_0^{\pi/2}\int_0^a r^5\cos\theta\sin^3\theta\,dr\,d\theta = \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{24}a^6$. Hence $(\overline{x}, \overline{y}) = \left(\frac{5}{32}\pi a, \frac{5}{8}a\right)$.

37. (a) The equation of the cone with the suggested orientation is $(h - z) = \frac{h}{a}\sqrt{x^2+y^2}$, $0 \le z \le h$. Then $V = \frac{1}{3}\pi a^2 h$ is the volume of one frustum of a cone; by symmetry $M_{yz} = M_{xz} = 0$; and

$$M_{xy} = \iint\limits_{x^2+y^2\le a^2}\int_0^{h-(h/a)\sqrt{x^2+y^2}} z\,dz\,dA = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} rz\,dz\,dr\,d\theta = \pi\int_0^a r\frac{h^2}{a^2}(a-r)^2\,dr$$

$$= \frac{\pi h^2}{a^2}\int_0^a (a^2r - 2ar^2 + r^3)\,dr = \frac{\pi h^2}{a^2}\left(\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{\pi h^2 a^2}{12}$$

Hence the centroid is $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{4}h\right)$.

(b) $$I_z = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} r^3\,dz\,dr\,d\theta = 2\pi\int_0^a \frac{h}{a}(ar^3 - r^4)\,dr = \frac{2\pi h}{a}\left(\frac{a^5}{4} - \frac{a^5}{5}\right) = \frac{\pi a^4 h}{10}$$

38. $1 \le z^2 \le 4 \;\Rightarrow\; 1/a^2 \le x^2 + y^2 \le 4/a^2$. Let $D = \{(x,y) \mid 1/a^2 \le x^2+y^2 \le 4/a^2\}$. $z = f(x,y) = a\sqrt{x^2+y^2}$, so $f_x(x,y) = ax(x^2+y^2)^{-1/2}$, $f_y(x,y) = ay(x^2+y^2)^{-1/2}$, and

$$A(S) = \iint_D \sqrt{\frac{a^2x^2 + a^2y^2}{x^2+y^2} + 1}\, dA = \iint_D \sqrt{a^2+1}\, dA = \sqrt{a^2+1}\, A(D)$$
$$= \sqrt{a^2+1}\left[\pi\left(\frac{2}{a}\right)^2 - \pi\left(\frac{1}{a}\right)^2\right] = \frac{3\pi}{a^2}\sqrt{a^2+1}$$

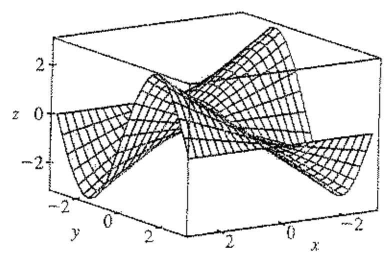
39. Let D represent the given triangle; then D can be described as the area enclosed by the x- and y-axes and the line $y = 2-2x$, or equivalently $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2-2x\}$. We want to find the surface area of the part of the graph of $z = x^2 + y$ that lies over D, so using Equation 15.6.3 we have

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (1)^2}\, dA = \int_0^1 \int_0^{2-2x} \sqrt{2+4x^2}\, dy\, dx$$
$$= \int_0^1 \sqrt{2+4x^2}\,\big[\,y\,\big]_{y=0}^{y=2-2x}\, dx = \int_0^1 (2-2x)\sqrt{2+4x^2}\, dx = \int_0^1 2\sqrt{2+4x^2}\, dx - \int_0^1 2x\sqrt{2+4x^2}\, dx$$

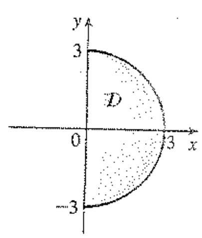
Using Formula 21 in the Table of Integrals with $a = \sqrt{2}$, $u = 2x$, and $du = 2\,dx$, we have $\int 2\sqrt{2+4x^2}\, dx = x\sqrt{2+4x^2} + \ln\!\left(2x + \sqrt{2+4x^2}\right)$. If we substitute $u = 2+4x^2$ in the second integral, then $du = 8x\,dx$ and $\int 2x\sqrt{2+4x^2}\, dx = \frac{1}{4}\int \sqrt{u}\, du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} = \frac{1}{6}(2+4x^2)^{3/2}$. Thus

$$A(S) = \left[x\sqrt{2+4x^2} + \ln\!\left(2x + \sqrt{2+4x^2}\right) - \tfrac{1}{6}(2+4x^2)^{3/2}\right]_0^1$$
$$= \sqrt{6} + \ln\left(2+\sqrt{6}\right) - \tfrac{1}{6}(6)^{3/2} - \ln\sqrt{2} + \tfrac{\sqrt{2}}{3} = \ln\tfrac{2+\sqrt{6}}{\sqrt{2}} + \tfrac{\sqrt{2}}{3}$$
$$= \ln\!\left(\sqrt{2}+\sqrt{3}\right) + \tfrac{\sqrt{2}}{3} \approx 1.6176$$

40. Using Formula 15.6.3 with $\partial z/\partial x = \sin y$, $\partial z/\partial y = x\cos y$, we get $S = \int_{-\pi}^{\pi}\int_{-3}^{3}\sqrt{\sin^2 y + x^2\cos^2 y + 1}\, dx\, dy \approx 62.9714$.



41.



$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\, dy\, dx = \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x(x^2+y^2)\, dy\, dx$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^3 (r\cos\theta)(r^2)\, r\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta \int_0^3 r^4\, dr$$
$$= \big[\sin\theta\big]_{-\pi/2}^{\pi/2}\left[\tfrac{1}{5}r^5\right]_0^3 = 2\cdot\tfrac{1}{5}(243) = \tfrac{486}{5} = 97.2$$

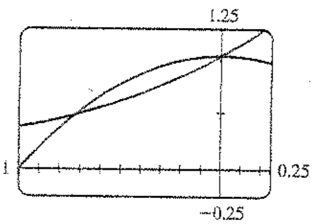
42. The region of integration is the solid hemisphere $x^2+y^2+z^2 \le 4$, $x \ge 0$.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^2 (\rho\sin\phi\sin\theta)^2\left(\sqrt{\rho^2}\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\int_0^2\rho^5\,d\rho$$

$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^2 = \left(\tfrac{\pi}{2}\right)\left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{32}{3}\right) = \tfrac{64}{9}\pi$$

43. From the graph, it appears that $1 - x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1 - x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is



$$\iint_D y^2\,dA \approx \int_{-0.71}^{0}\int_{e^x}^{1-x^2} y^2\,dy\,dx$$

$$= \tfrac{1}{3}\int_{-0.71}^{0}\left[(1-x^2)^3 - e^{3x}\right]dx$$

$$= \tfrac{1}{3}\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{1}{3}e^{3x}\right]_{-0.71}^{0} \approx 0.0512$$

44. Let the tetrahedron be called T. The front face of T is given by the plane $x + \frac{1}{2}y + \frac{1}{3}z = 1$, or $z = 3 - 3x - \frac{3}{2}y$, which intersects the xy-plane in the line $y = 2 - 2x$. So the total mass is

$m = \iiint_T \rho(x,y,z)\,dV = \int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2}(x^2+y^2+z^2)\,dz\,dy\,dx = \frac{7}{5}$. The center of mass is

$(\overline{x},\overline{y},\overline{z}) = \left(m^{-1}\iiint_T x\rho(x,y,z)\,dV, m^{-1}\iiint_T y\rho(x,y,z)\,dV, m^{-1}\iiint_T z\rho(x,y,z)\,dV\right) = \left(\frac{4}{21},\frac{11}{21},\frac{8}{7}\right)$.

45. (a) $f(x,y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,3]\times[0,2]$, we can say

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^3\int_0^2 C(x+y)\,dy\,dx$$

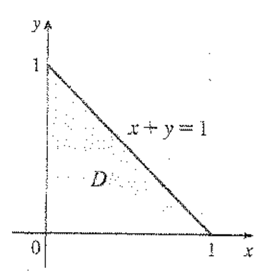
$$= C\int_0^3\left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=2}dx = C\int_0^3(2x+2)\,dx = C\left[x^2+2x\right]_0^3 = 15C$$

Then $15C = 1 \Rightarrow C = \frac{1}{15}$.

(b) $P(X \le 2, Y \ge 1) = \int_{-\infty}^{2}\int_1^{\infty} f(x,y)\,dy\,dx = \int_0^2\int_1^2 \frac{1}{15}(x,y)\,dy\,dx = \frac{1}{15}\int_0^2\left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=2}dx$

$$= \tfrac{1}{15}\int_0^2\left(x + \tfrac{3}{2}\right)dx = \tfrac{1}{15}\left[\tfrac{1}{2}x^2 + \tfrac{3}{2}x\right]_0^2 = \tfrac{1}{3}$$

(c) $P(X + Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y\le 1) = \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x}\tfrac{1}{15}(x+y)\,dy\,dx$$

$$= \tfrac{1}{15}\int_0^1\left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x}dx$$

$$= \tfrac{1}{15}\int_0^1\left[x(1-x) + \tfrac{1}{2}(1-x)^2\right]dx$$

$$= \tfrac{1}{30}\int_0^1(1-x^2)\,dx = \tfrac{1}{30}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{45}$$