EXTRA EXAM

Math 233 Final exam May 2, 2013

Name: Solutions
Please print above

Course: *Math 233*

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- To see your exam score, go to the math department homepage at www.math.wustl.edu and use the link to 'Exam/hw/quiz scores' under 'Resources'.
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For more information about your exam, contact your instructor or the math department office in Cupples I, room 100.

Math 233 - FINAL - Spring 2013

May 2, 2013

NAME:

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STUDENT ID NUMBER:

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General instructions: This exam has 16 questions, each worth the same amount. Check that no pages are missing and notify your proctor if you detect any problems with your copy of the exam. Mark your ID number on the six blank lines on the top of your answer card, using one line for each digit. Print your name on the top of the card. Choose the answer that is closest to the solution and mark your answer card with a PENCIL by shading in the correct box. You may use a 3×5 card with notes and any calculator that does not have graphing functions. GOOD LUCK!

1. Find a vector parallel to $\mathbf{a} \times \mathbf{b}$, when $\mathbf{a} = \langle -1, 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 1, 0 \rangle$.

(A) $\langle -1, 1, 2 \rangle$
(B) $\langle -3, -1, 0 \rangle$
(C) $\langle -66, 22, 0 \rangle$
(D) $\langle -34, 58, -100 \rangle$
(E) $\langle -23, -46, 99 \rangle$
(F) $\langle -66, -22, 0 \rangle$
(G) $\langle -12, 36, -24 \rangle$
(H) $\langle -12, -42, -84 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= i(-2) - j(-6) + k(-1-3)$$

$$= \langle -2, 6, -4 \rangle$$

$$6\langle -2, 6, -4 \rangle = \langle -12, 36, -24 \rangle$$

2. Give the x-coordinate of the point of intersection of the plane $x + 2y + z = 6$ and the line through the points $(1, 0, 1)$ and $(2, -1, 1)$.

(A) -3

(B) -2

(C) -1

(D) 0

(E) 1

(F) 2

(G) 3

(H) 4

Find equation of line:

$$\vec{r}(t) = (1-t)\langle 1,0,1\rangle + t\langle 2,-1,1\rangle$$

$$= \langle 1+t, -t, 1\rangle$$

So
$$(*)\begin{cases} x(t) = 1+t \\ y(t) = -t \\ z(t) = 1 \end{cases}$$

Plug into equation of plane $\quad x + 2y + z = 6$

$$(1+t) + 2(-t) + 1 = 6$$

$$-t + 2 = 6$$

$$t = -4$$

Plug back into $(*)$: $\quad x(t) = 1 - 4 = -3$

3. Find the length of the curve given by $\mathbf{r}(t) = \langle 2\,sint, \sqrt{5}t, 2\,cost \rangle$, $0 \le t \le 2$.

(A) 3

(B) $2\sqrt{5}$

(C) $3\sqrt{5}$

(D) 6

(E) $12\sqrt{3}$

(F) 15

(G) 20

(H) 25

$\vec{r}\,'(t) = \langle 2\cos t, \sqrt{5}, -2\sin t \rangle$

$|\vec{r}\,'(t)| = \sqrt{4\cos^2 t + 5 + 4\sin^2 t} = \sqrt{9} = 3$

$\therefore \quad L = \int_0^2 3\,dt = 3t\Big|_0^2 = 6$

4. Describe the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2 - 2x - 4y + 8z - 15 \quad = c$$

(A) planes with normal $\langle -2, -4, 8 \rangle$

(B) planes with normal $\langle -1, 1, 1 \rangle$

(C) planes with normal $\langle -1, -1, -1 \rangle$

(D) spheres with center $(0, 0, 0)$

(E) spheres with center $(-2, 4, 8)$

(F) spheres with center $(-1, 1 - 1)$

(G) spheres with center $(-1, -3, -9)$

(H) spheres with center $(1, 2, -4)$

$$(x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 + 8z + 16) = 15 + c + 21$$

$$(x-1)^2 + (y-2)^2 + (z+4)^2 = 36 + c$$

$\therefore$ spheres with centre $(1, 2, -4)$

5. Suppose $z = f(x, y)$, where f is differentiable, and $x = g(t), y = h(t)$, where g and h differentiable. Suppose also that $g(1) = -3, g'(1) = 2, h(1) = 5, h'(1) = 4$, $f_x(1, 1) = 2\pi$, $f_x(-3, 5) = -\pi$, $f_y(1, 1) = -2\pi$, and $f_y(-3, 5) = 6\pi$. Find $\frac{dz}{dt}$ when $t = 1$.

(A) 22π

(B) 5π

(C) -4π

(D) -5π

(E) -8π

(F) 10

(G) -12

(H) 20

$$\begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & t & \end{tikzcd}$$

$$\vec{r}(1) = (g(1), h(1)) = (-3, 5)$$

$$\left.\frac{dz}{dt}\right|_{t=1} = \left.\frac{\partial z}{\partial x}\right|_{r(1)} \left.\frac{dx}{dt}\right|_1 + \left.\frac{\partial z}{\partial y}\right|_{r(1)} \left.\frac{dy}{dt}\right|_1$$

$$= f_x(-3,5)\, g'(1) + f_y(-3,5)\, h'(1)$$

$$= (-\pi)(2) + (6\pi)(4)$$

$$= 22\pi$$

6. Find the maximum rate of change of $f(x, y) = ye^{xy}$ at the point $(0, 2)$.

(A) 2

(B) e

(C) 4

(D) 5

(E) $4e$

(F) $4\sqrt{5}$

(G) $\sqrt{17}$

(H) 9.5

max rate of change $= |\nabla f|$

$\nabla f = \langle f_x, f_y \rangle = \langle y^2 e^{xy}, e^{xy} + xy e^{xy} \rangle$

$\nabla f(0, 2) = \langle 4, 1 \rangle$

$\therefore \quad |\nabla f(0,2)| = \sqrt{4^2 + 1^2} = \sqrt{17}$

7. Suppose $f(x, y)$ is function with continuous second partial derivatives. Suppose also that $f_x(0,0) = f_y(0,0) = 0$ and $f_x(1,1) = f_y(1,1) = 1$ and that

$$f_{xx}(0,0) = f_{xx}(1,1) = \pi, \; f_{yy}(0,0) = f_{yy}(1,1) = 1, \; f_{xy}(0,0) = 2, \; f_{xy}(1,1) = 1.$$

What can you conclude?

(A) $(0,0)$ is a local min and $(1,1)$ is a local max
(B) $(1,1)$ is a local min and $(0,0)$ is a local max
(C) both $(0,0)$ and $(1,1)$ are saddle points
(D) $(0,0)$ is a local max, no conclusion about $(1,1)$
(E) $(0,0)$ is a local min, no conclusion about $(1,1)$
(F) $(1,1)$ is a local max, no conclusion about $(0,0)$
(G) $(1,1)$ is a local min, no conclusion about $(0,0)$
(H) $(0,0)$ is a saddle point, $(1,1)$ is not a local extremum

$(0,0)$ = critical point

$(1,1) \neq$ critical point
$\therefore$ NOT a local extremum

At $(0,0)$:

$$D = \begin{vmatrix} \pi & 2 \\ 2 & 1 \end{vmatrix} = \pi - 4 < 0$$

$\therefore$ saddle point at $(0,0)$

8. Find the volume of the solid under the surface $z - xy = 1$ and above the bounded region in the upper half-plane enclosed by $x = y^2$, $y = 0$, and $x = 4$.

(A) 40/3

(B) 39/2

(C) 97/7

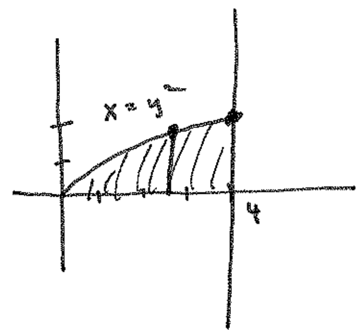
(D) $4\sqrt{2}$

(E) 15

(F) 16

(G) 17

(H) 18



$$\int_0^4 \int_0^{\sqrt{x}} (xy + 1)\, dy\, dx$$

$$\int_0^4 \left(\frac{1}{2} x y^2 \Big|_0^{\sqrt{x}} + y \Big|_0^{\sqrt{x}} \right) dx$$

$$= \int_0^4 \left(\frac{1}{2} x^2 + x^{1/2} \right) dx$$

$$= \frac{1}{6} x^3 + \frac{2}{3} x^{3/2} \Big|_0^4$$

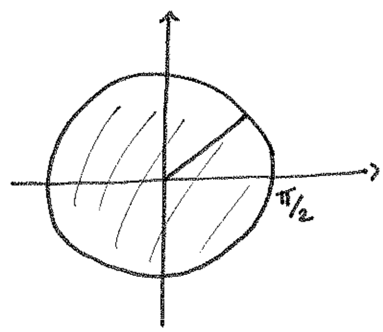
$$= 16$$

9. Evaluate

$$\iint_D cos\sqrt{x^2+y^2}\,dA$$

when D is the disk bounded by the circle $x^2+y^2=\pi^2/4$.

(A) 0
(B) $\pi^2-2\pi$
(C) $\frac{\pi}{2}(1-e^{-4})$
(D) $2\pi(2sin\,2+cos\,2)$
(E) $2\pi(2sin\,2+cos\,2-1)$
(F) $\pi^5/16$
(G) $\pi^2/4$
(H) $sin(\pi^2/4)$



Use polar coordinates :

$$\underbrace{\int_0^{2\pi}\int_0^{\pi/2}(\cos r)\,r\,dr}\,d\theta$$

$$\int \underbrace{r}_{u}\underbrace{\cos r\,dr}_{dv} = \underbrace{r\sin r}_{uv} - \underbrace{\int \sin r\,dr}_{\int v\,du} = r\sin r + \cos r + K$$

$$\therefore \int_0^{\pi/2} r\cos r\,dr = r\sin r + \cos r\Big|_0^{\pi/2} = \pi/2 - 1$$

$$\therefore \int_0^{2\pi}\left(\frac{\pi}{2}-1\right)d\theta = \left(\frac{\pi}{2}-1\right)\theta\Big|_0^{2\pi} = 2\pi\left(\frac{\pi}{2}-1\right)$$

$$= \pi^2 - 2\pi$$

10. Evaluate

$$\int_0^1 6y\,dx$$

(A) 0

(B) -3

(C) 3

(D) -6

(E) 6y

(F) -9x

(G) 9y

(H) 10x

$$\int_0^1 6y\,dx = 6 \cdot xy \Big|_{x=0}^{x=1}$$

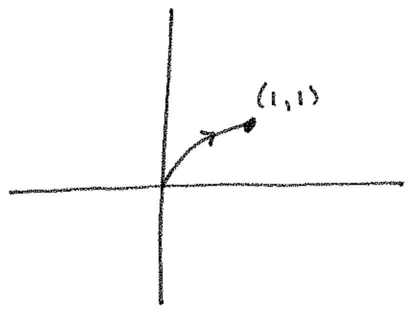
$$= 6y$$

11. Evaluate

$$\int_C 6y\,ds$$

when C is the parabola $x = y^2$ from $(0,0)$ to $(1,1)$.

(A) $0.5(5\sqrt{5} - 1)$
(B) $(5\sqrt{5} - 5)/6$
(C) $\sqrt{5} - 2$
(D) $0.5\sqrt{5}$
(E) 0
(F) $1/2$
(G) $2/3$
(H) $3/4$



$$x = y^2$$

$$\therefore \quad \vec{r}(t) = \langle t^2, t \rangle \qquad 0 \le t \le 1$$

$$f(\vec{r}(t)) = 6t$$

$$|r'(t)| = |\langle 2t, 1 \rangle| = \sqrt{4t^2+1}$$

$$\therefore \quad \int_C 6y\,ds = \int_0^1 6t\sqrt{4t^2+1}\,dt \qquad u = 4t^2+1, \quad du = 8t\,dt$$

$$= \frac{6}{8}\int_1^5 u^{1/2}\,du$$

$$= \frac{3}{4} \cdot \frac{2}{3}\, u^{3/2}\Big|_1^5$$

$$= \frac{1}{2}(5\sqrt{5} - 1)$$

12. Evaluate

$$\int_C 6y\,dx$$

when C is the parabola $x = y^2$ from $(0,0)$ to $(1,1)$.

(A) 0

(B) -3

(C) 3

(D) -4

(E) 4

(F) -9

(G) 9

(H) 10

$\vec{r}(t) = \langle t^2, t \rangle \qquad 0 \le t \le 1$

$x'(t) = 2t$

$\int_C 6y\,dx = \int_0^1 (6t)(2t)\,dt$

$= 12 \int_0^1 t^2\,dt$

$= 4t^3 \Big|_0^1$

$= 4$

13. Evaluate

$$\int_C y\,dx + x^2\,dy$$

when C is the line segment from $(-2,-1)$ to $(0,2)$.

(A) -3.5

(B) 3.5

(C) -4

(D) 4

(E) -4.5

(F) 4.5

(G) -5

(H) 5

$$F(x,y) = \langle y, x^2 \rangle$$

$$\vec{r}(t) = (1-t)(-2,-1) + t(0,2) \qquad 0 \le t \le 1$$

$$= \langle \underbrace{2t-2}_{x}, \underbrace{3t-1}_{y} \rangle$$

$$\vec{r}\,'(t) = \langle 2, 3 \rangle$$

$$F(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle 3t-1, (2t-2)^2 \rangle \cdot \langle 2, 3 \rangle$$

$$= 2(3t-1) + 3(4t^2 - 8t + 4)$$

$$= 12t^2 - 18t + 10$$

$\therefore$

$$\int_C y\,dx + x^2 dy = \int_0^1 (12t^2 - 18t + 10)\,dt$$

$$= 4t^3 - 9t^2 + 10t \Big|_0^1$$

$$= 5$$

14. Evaluate

$$\int_C y \sin z \, ds$$

when C is the circular helix given by

$$x(t) = \cos t, \;\; y(t) = \sin t, \;\; z(t) = t, \quad 0 \leq t \leq 4\pi.$$

(A) $\sqrt{2}\pi$

(B) $2\sqrt{2}\pi$

(C) $2\pi - 1$

(D) $3\pi - 1$

(E) $\sqrt{3}\pi$

(F) $3\sqrt{2}$

(G) $2\sqrt{3}$

(H) 0

$$\int_0^{4\pi} f(\vec{r}(t)) \, |\vec{r}\,'(t)| \, dt$$

$$f(\vec{r}(t)) = \sin^2 t$$

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad 0 \leq t \leq 4\pi$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{2}$$

$$\therefore \int_C y \sin z \, ds = \sqrt{2} \int_0^{4\pi} \sin^2 t \, dt$$

$$= \sqrt{2} \int_0^{4\pi} \frac{1 - \cos 2t}{2} \, dt$$

$$= \frac{\sqrt{2}}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{4\pi}$$

$$= \frac{\sqrt{2}}{2} (4\pi) = 2\sqrt{2}\pi$$

15. Find the work done by the force field

$$\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$$

in moving a particle along the curve

$$\mathbf{r}(t) = cos t\,\mathbf{i} + cos^2\,t\,\mathbf{j} + cos^5\,(2t)\,\mathbf{k}, \quad 0 \le t \le \pi/2.$$

(A) 0
(B) -1
(C) 1
(D) -2
(E) 2
(F) 3
(G) -3
(H) $\pi/2$

F is conservative :

$$F(x,y,z) = \langle yz , xz , xy \rangle = \nabla f$$

for $f(x,y,z) = xyz$

$$\therefore \int_0^{\pi/2} \vec{F} \cdot d\vec{r} = f(\vec{r}(\tfrac{\pi}{2})) - f(\vec{r}(0))$$

$$= f(0,0,-1) - f(1,1,1)$$

$$= 0 - 1$$

$$= -1$$

16. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

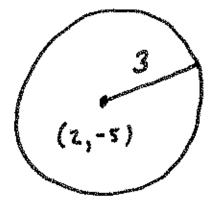
when

$$\mathbf{F}(x, y) = \langle \underbrace{y - cos\, y}_{P}, \underbrace{x sin\, y}_{Q} \rangle$$

and C is the circle

$$(x-2)^2 + (y+5)^2 = 9$$

oriented clockwise.



(A) 0
(B) 1
(C) -2π
(D) $5e$
(E) $11\sqrt{2}$
(F) 9π
(G) $-10\sqrt{3}$
(H) 30

Use Green's Thm :

$$\int_C F \cdot dr = - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= - \iint_D \left(\sin y - (1 + \sin y) \right) dA$$

$$= - \iint_D - dA$$

$$= \text{Area}(D)$$

$$= \pi \cdot 3^2$$

$$= 9\pi$$