Math 131

Midterm Examination 1 Solutions – February 8, 2009

General Instructions: You may use a simple calculator that is not graph-
ing or programmable. You may have a 3x5 card, but no other notes.

Part I (72 points): For each of the following 12 multiple choice prob-
lems, mark your answer on the answer card. For Part I, only the answer
on the card will be graded. Each problem is worth 6 points.

1. Compute

$$\lim_{x \to 0} x^2 + 1$$

   (a) Undefined/doesn't exist

   (b) 0

   (c) 1

   (d) 2

   (e) $\sqrt{3}$

   (f) -1

   (g) -2

   (h) $-\sqrt{3}$

   Since $x^2 + 1$ is continuous everywhere, we plug in $0^2 + 1 = 1$. The
   answer is C.

2. Which of the following functions are continuous at $x = 0$?
   1. $\sin \dfrac{1}{x}$ 2. $\dfrac{1}{x^2}$ 3. $\dfrac{1}{x^2 + 1}$

   (a) None of the above

   (b) Only 1.

   (c) Only 2.

   (d) Only 1 and 2.

   (e) Only 3.

(f) Only 1 and 3.

(g) Only 2 and 3.

(h) All of the above

The first two functions are not defined at $x = 0$, so certainly not continuous there. The third function is defined at $x = 0$, and rational functions are continuous everywhere that they are defined. The answer is E.

3. Find $\lim\limits_{x \to 0+} \cos \dfrac{1}{x}$.

(a) Undefined/doesn't exist

(b) 0

(c) 1

(d) 2

(e) ∞

(f) -1

(g) -2

(h) $-\infty$

The function $\cos \frac{1}{x}$ oscillates infinitely as $x \to 0$, so the limit does not exist. The answer is A.

4. Compute
$$\lim_{x \to 3} \frac{3 - x}{\sqrt{x} - \sqrt{3}}$$

(a) Undefined/doesn't exist

(b) 0

(c) 1

(d) -1

(e) $\sqrt{3}$

(f) $-\sqrt{3}$

(g) $2\sqrt{3}$

(h) $-2\sqrt{3}$

When we try plugging in 3, we get something of the form "$\frac{0}{0}$". We cancel the zeros, using the conjugate technique from class:

$$\lim_{x\to 3} \frac{3-x}{\sqrt{x}-\sqrt{3}} \cdot \frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} = \lim_{x\to 3} \frac{(3-x)\cdot(\sqrt{x}+\sqrt{3})}{x-3}$$

$$= \lim_{x\to 3} -(\sqrt{x}+\sqrt{3}).$$

Since $-(\sqrt{x}+\sqrt{3})$ is continuous, we plug in 3 to get $-2\sqrt{3}$. The correct answer is H.

Since sign errors are especially easy to make, I gave half credit for answer G.

5. Consider the function $g(x) = \dfrac{x^2-1}{|x+1|}$. Find $\lim\limits_{x\to -1+} g(x)$.

   Hint: Remember the piecewise definition of $|x+1|$!

   (a) Undefined/doesn't exist

   (b) 0

   (c) 1

   (d) 2

   (e) ∞

   (f) -1

   (g) -2

   (h) $-\infty$

We follow the hint, and write

$$g(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if } x+1 \geq 0 \\ \frac{x^2-1}{-(x+1)} & \text{if } x+1 < 0 \end{cases}$$

$$= \begin{cases} x-1 & \text{if } x > -1 \\ 1-x & \text{if } x < -1 \\ \text{undefined} & \text{if } x = -1 \end{cases}.$$

Since the limit is as $x \to -1+$, we plug into the function for $x \geq -1$, to get -2. The answer is G.

6. Consider the function $g(x) = \dfrac{x^2 - 1}{|x + 1|}$. Find $\displaystyle\lim_{x \to -1} g(x)$

    (a) Undefined/doesn't exist
    (b) 0
    (c) 1
    (d) 2
    (e) ∞
    (f) -1
    (g) -2
    (h) $-\infty$

    See the answer for Problem 5. The limit as $x \to -1-$ we find by plugging into $1 - x$, for which we get 2. Since $2 \neq -2$, the left and right limits do not agree, and the (two-sided) limit is undefined.

    The answer is A.

7. Let
$$f(x) = \begin{cases} -1 & \text{if } x < -5 \\ x^2 & \text{if -5} \leq \text{x} < 0 \\ x^3 & \text{if 0} < \text{x} \leq 5 \\ x^2 + 100 & \text{if } x > 5 \end{cases}.$$
    Find $\displaystyle\lim_{x \to 0} f(x)$.

    (a) Undefined/doesn't exist
    (b) 0
    (c) 1
    (d) 2
    (e) ∞
    (f) -1

(g) -2

(h) $-\infty$

We calculuate $\lim_{x\to0+} f(x) = \lim_{x\to0+} x^3 = 0$, and $\lim_{x\to0-} f(x) = \lim_{x\to0-} x^2 = 0$. Since the left and right limits agree, the two-sided limit is the same. The answer is B.

8. Let

$$f(x) = \begin{cases} -1 & \text{if } x < -5 \\ x^2 & \text{if -5}\leq\text{x}<0 \\ x^3 & \text{if } 0<\text{x}\leq5 \\ x^2 + 100 & \text{if } x > 5 \end{cases},$$

the same as in the last problem. Where is f continuous?

(a) Everywhere.

(b) Everywhere except -5, 0, 1, and 5.

(c) Everywhere except -5, 0, and 5.

(d) Everywhere except -5, and 1.

(e) Everywhere except -5.

(f) Everywhere except -5, and 0.

(g) Everywhere except 5.

(h) Everywhere except 0.

The function is clearly continuous everywhere except possibly -5, 0, and 5. We check each point.

At -5: $\lim_{x\to-5-} f(x) = \lim_{x\to-5-} -1 = -1$, while $\lim_{x\to-5+} f(x) = \lim_{x\to-5+} x^2 = 25$. Since these disagree, the (two-sided) limit does not exist, so the function is not continuous at -5.

At 0: The function is not defined at 0, so not continuous at 0.

At 5: $\lim_{x\to5-} f(x) = \lim_{x\to5-} x^3 = 125$, while $\lim_{x\to5+} f(x) = \lim_{x\to5+} x^2 + 100 = 125$. Since the left and right limits agree, the two-sided limit is also 125. Also $f(5) = 5^3 = 125$, the same as the limit. Thus the function is continuous at 5.

The answer is F. Since the behavior at 0 was a bit tricky, I gave half credit for E.

9. If you take the graph of $f(x) = x^3$, and first shift it left by 4 units, then expand it vertically by a factor of two, what function do you get?

   (a) $2x^3 + 4$

   (b) $2x^3 - 4$

   (c) $2(x + 4)^3$

   (d) $2(x - 4)^3$

   (e) $\frac{1}{2}(x + 4)^3$

   (f) $\frac{1}{2}(x - 4)^3$

   (g) $\frac{1}{2}x^3 + 4$

   (h) $\frac{1}{2}x^3 - 4$

   Shifting left replaces x by $x + 4$. Expanding vertically multiplies by 2. The answer is C.

10. Find $\lim\limits_{x \to 0} \dfrac{x^2 - e^x \sin x}{\cos^2 x}$.

   (a) Undefined/doesn't exist

   (b) 0

   (c) 1

   (d) 2

   (e) ∞

   (f) -1

   (g) -2

   (h) $-\infty$

   Since the function is defined at 0, and built from x^2, e^x, $\sin x$, and $\cos x$ by subtraction, multiplication, and division, it is continuous. Thus, we plug in $x = 0$: $\frac{0^2 - 1 \cdot 0}{1^2} = 0$. The answer is B.

11. Find $\lim\limits_{x \to 5} \dfrac{(3x - 15) \cdot \sin(x - 5)}{2(x - 5)^2}$

   (a) Undefined
   (b) $-\frac{3}{2}$
   (c) $\frac{2}{3}$
   (d) $\frac{6}{2}$
   (e) $\frac{3}{2}$
   (f) $\frac{9}{2}$
   (g) $\frac{1}{3}$
   (h) $-\frac{9}{2}$

   We rewrite as

   $$\lim_{x \to 5} \frac{3}{2} \cdot \frac{(x - 5)\sin(x - 5)}{(x - 5)^2} = \lim_{x \to 5} \frac{3}{2} \cdot \frac{\sin(x - 5)}{(x - 5)},$$

   and since $\frac{\sin(x-5)}{(x-5)}$ is $\frac{\sin x}{x}$ shifted to the right by 5, the limit $\lim_{x \to 5} \frac{\sin(x-5)}{(x-5)}$ is 1. Thus, our overall answer is $\frac{3}{2}$, or E.

12. Which of the below is equivalent with the statement "f is continuous at a"?

   (a) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - a| < \delta \implies |f(x) - L| < \epsilon$.
   (b) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$.
   (c) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(L)| < \epsilon$.
   (d) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < x - a < \delta \implies |f(x) - L| < \epsilon$.
   (e) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < x - a < \delta \implies |f(x) - f(a)| < \epsilon$.
   (f) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < x - a < \delta \implies |f(x) - f(L)| < \epsilon$.

(g) for all $\epsilon < 0$ there exists a $\delta < 0$ such that $|x - a| < \delta \implies |f(x) - L| < \epsilon$.

(h) Undefined/doesn't exist

The answer A is essentially the definition of $\lim_{x \to a} f(x) = L$ (the definition of limit also has $0 < |x - a|$). However, we were asked about the definition of continuous function, which means $\lim_{x \to a} f(x) = f(a)$. When we plug $f(a)$ in for L, we get B.

Note: D is essentially the definition of right-sided limit, while E is the definition of right-sided continuity ($\lim_{x \to a+} f(x) = f(a)$).

The answer is B. Two-thirds of the class put A – I gave half credit for this answer.

Part II (28 points): In each of the following problems, show your work clearly in the space provided. Partial credit will be given, and a correct answer without supporting work may not receive credit.

1. Rates of change and tangent lines
   Let $f(x) = x^3$.

   (a) (2 points) Graph $f(x)$, and on the same axis, the straight line from $(0, 0)$ to (h, h^3).
   (You might remember that this line is called a secant.)
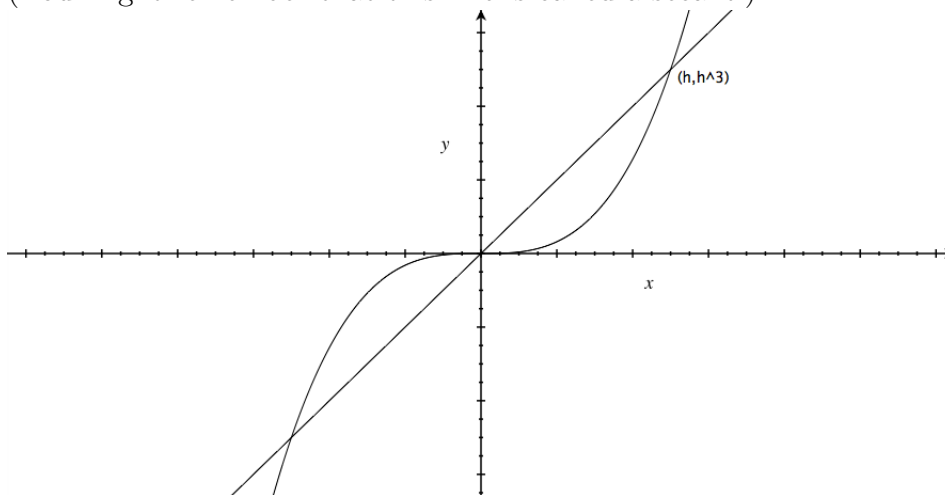
(b) (4 points) Find the average rate of change in f on the interval $[0, h]$.

The average rate of change is $\frac{\Delta y}{\Delta x}$, that is, the slope of the line in part (a). We calculate this to be

$$\frac{f(h) - f(0)}{h - 0} = \frac{h^3 - 0}{h - 0} = \frac{h^3}{h}.$$

(c) (5 points) By taking a limit, find the instantaneous rate of change of f at 0.

The instantaneous rate of change is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, that is,

$$\lim_{h \to 0} \frac{h^3}{h} = \lim_{h \to 0} h^2 = 0.$$

2. Bounds

(a) (4 points) Find a bound for $|2 \sin \frac{1}{x^2} + 3 \cos \frac{1}{x^2}|$.

Following the handout, we break it up:

$$\begin{aligned} |2 \sin \frac{1}{x^2} + 3 \cos \frac{1}{x^2}| &\leq |2 \sin \frac{1}{x^2}| + |3 \cos \frac{1}{x^2}| \\ &= 2|\sin \frac{1}{x^2}| + 3|\cos \frac{1}{x^2}| \\ &\leq 2 + 3 = 5, \end{aligned}$$

where the last step is because $\sin *$ and $\cos *$ are bounded by 1. Our bound is 5.

(b) (5 points) Using the sandwich theorem, explain why $\lim_{x \to 0} x^3 \sin \frac{1}{x^2} = 0$.

Similarly to part (a), we know that

$$-1 \leq \sin \frac{1}{x^2} \leq 1,$$

hence

$$-|x^3| \leq x^3 \sin \frac{1}{x^2} \leq |x^3|.$$

Since $\lim_{x \to 0} x^3 = \lim_{x \to 0} |x^3| = \lim_{x \to 0} -|x^3| = 0$, $x^3 \sin \frac{1}{x^2}$ is sandwiched, and also has limit 0.

3. Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$.

(a) (3 points) For what values of x is f continuous?

All values. The function $\frac{\sin x}{x}$ is continuous at all values except 0, and so $f(x)$ is continuous at all such values. Zero is explained in part (b).

(b) (5 points) Explain carefully why the above $f(x)$ is continuous or not continuous at $x = 0$.

For f to be continuous at 0, we must have $\lim_{x \to 0} f(x) = f(0)$. On the one hand, $f(0) = 1$. On the other,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

These are equal, so the function is indeed continuous at 0.

The graph of $f(x)$ is as follows: