Math 131
Midterm Examination 2 Solutions – March 13, 2009

General Instructions: You may use a simple calculator that is not graphing or programmable. You may have a 3x5 card, but no other notes.

Part I (73 points): For each of the following 17 problems, mark your answer on the answer card. For Part I, only the answer on the card will be graded.

Problems 1-13: Multiple choice. Each problem is worth 5 points.

1. Let $f(x) = \cos x - 12e^x$. Find $f'(x)$.

   (a) $-\sin x - 12e^x$
   (b) $\sin x - 12e^x$
   (c) $-\sin x + 12e^x$
   (d) $\sin x + 12e^x$
   (e) $-\sin x - 12xe^{x-1}$
   (f) $\sin x - 12xe^{x-1}$
   (g) does not exist
   (h) ∞

   A quick calculation gives the answer to be A.

2. Compute $\dfrac{d}{dx}\left[(x^3-1)^{2009}\right]$.

   (a) $2009(x^3-1)^{2008}$
   (b) $2009(x^3-1)^{2008} \cdot 3x^2$
   (c) $2009(x^3-1)^{2009}$
   (d) $2008(x^3-1)^{2009}$
   (e) $2008(x^3-1)^{2009} \cdot 3x^2$
   (f) $2009(x^3-1)^{2008} \cdot (3x^2-1)$

(g) does not exist

(h) ∞

We apply the chain rule: $\frac{d}{dx}(x^3-1)^{2009} = 2009(x^3-1)^{2008} \cdot \frac{d}{dx}\left[x^3-1\right]$ =answer B.

3. Let $f(x) = x^e \cdot e^x$. Calculate $f'(x)$.

   (a) $ex^{e-1}e^x$

   (b) $x^{e+1}e^{x-1}$

   (c) $ex^{e-1}e^x + x^e e^x$

   (d) $x^e e^x + x^{e+1}e^x$

   (e) $x^\pi e^x + x^e e^x$

   (f) $\pi x^{\pi-1}e^x$

   (g) does not exist

   (h) ∞

We recall that the derivative of e^x is e^x, while that of x^e is ex^{e-1}. We then apply the product rule, and find that the answer is C.

4. Find $\lim_{x \to 2} \dfrac{1}{x-2}$.

   (a) ∞

   (b) $-\infty$

   (c) does not exist

   (d) e

   (e) π

   (f) 17

   (g) 21

   (h) Rabbit.

We notice that $\frac{1}{x-2}$ is $\frac{1}{x}$ shifted horizontally by 2. Thus, $\lim_{x\to 2+}\frac{1}{x-2} = \infty$ and $\lim_{x\to 2-}\frac{1}{x-2} = -\infty$, hence the two-sided limit DNE: answer C.

5. Compute $\dfrac{d}{dx}\left[e^{-x^2\cos 3x}\right]$.

   (a) $e^{-x^2\cos 3x}$

   (b) $e^{-x^2\cos 3x}\cdot x^2\sin 3x$

   (c) $e^{-x^2\cos 3x}\cdot(-2x\cos 3x + 3x^2\sin 3x)$

   (d) $e^{-x^2\cos 3x}\cdot(-6x)\cdot\sin 3x)$

   (e) $e^{-x^2\cos 3x}\cdot 6x\cdot\sin 3x$

   (f) $-x^2\cos 3x e^{-x^2\cos 3x-1}$

   (g) does not exist

   (h) ∞

We apply the chain rule, then the product rule, then the chain rule.

$$\begin{aligned}
\frac{d}{dx}e^{-x^2\cos 3x} &= e^{-x^2\cos 3x}\cdot\frac{d}{dx}\left[-x^2\cos 3x\right]\\
&= e^{-x^2\cos 3x}\cdot\left(-2x\cos 3x + x^2\cos 3x\cdot\frac{d}{dx}[3x]\right)\\
&= e^{-x^2\cos 3x}\cdot(-2x\cos 3x + x^2\cos 3x\cdot 3),\ \text{answer C.}
\end{aligned}$$

6. Compute $\dfrac{d^2}{dx^2}\left[\dfrac{3x}{(1-x)}\right]$.

   (a) $\dfrac{6}{(1-x)^3}$

   (b) $\dfrac{-3}{2(1-x)}$

   (c) $\dfrac{3}{(1-x)^4}$

   (d) $\dfrac{-12+18x}{(1-x)^3}$

(e) $\frac{0}{0}$
(f) -3
(g) does not exist
(h) ∞

We use the quotient rule, and calculate the first derivative to be

$$\frac{3 \cdot (1-x) - 3x \cdot (-1)}{(1-x)^2} = \frac{3}{(1-x)^2}.$$

We then use either the quotient rule or the chain rule to calculate the derivative to be answer A. (Don't forget to use the chain rule in differentiating $(1-x)^2$!)

7. Find the tangent line to the graph of $y = x^3 - x - 2$ at $x = 2$.

    (a) $y = 11x$
    (b) $y - 11 = 2(x - 4)$
    (c) $y - 4 = 9(x - 2)$
    (d) $y = 11x + 4$
    (e) $y = 11x - 18$
    (f) $y = 2x$
    (g) $x = 2$ (vertical tangent)
    (h) does not exist

    We calculate the derivative to be $y' = 3x^2 - 1$. At $x = 2$, this gives the slope of the tangent line to be $3 \cdot 2^2 - 1 = 11$. In point slope form, the tangent line is $y - 4 = 11(x - 2)$, which is equivalent to answer E.

8. At what values of x does the function $g(x) = x^3 - 3x - 1$ have a horizontal tangent?

    (a) At 1, 0, and -1 only.
    (b) At 0 and -1 only.

(c) At 1 and 0 only

(d) At 1 and -1 only.

(e) At -1 only.

(f) At 1 only.

(g) At 0 only

(h) $g(x)$ has no horizontal tangents.

A function has a horizontal tangent when its derivative is zero. We calculate

$$g'(x) = 3x^2 - 3 = 3 \cdot (x^2 - 1) = 3(x+1)(x-1).$$

The derivative is thus zero at $x = -1$ and $x = 1$, answer D. (But $g'(0) = -3!$)

9. Find $\lim\limits_{x \to -\infty} \dfrac{3x^3 - 2x}{15x^3 + 1}$

(a) $\dfrac{18}{15}$

(b) $\dfrac{3}{45}$

(c) 0

(d) ∞

(e) $\dfrac{3 - 2x}{15 + 1}$

(f) $\dfrac{3}{15}$

(g) $-\infty$

(h) does not exist

We calculate

$$\lim_{x \to -\infty} \frac{3x^3 - 2x}{15x^3 + 1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x \to -\infty} \frac{3 - \frac{2}{x^2}}{15 + \frac{1}{x^3}} = \frac{3 - 0}{15 + 0} = \frac{3}{15}$$

or answer F.

10. Let y be implicitly related to x by the equation $y^3+xy+x^3 = 12$. Find an implicit relationship for $\dfrac{dy}{dx}$.

   (a) $y^3 + y + 3x^2 = 0$
   (b) $3y^2\dfrac{dy}{dx} + \dfrac{dy}{dx} + 3x^2 = 0$
   (c) $3y^2\dfrac{dy}{dx} + y + x + 3x^2 = 0$
   (d) $3y^2\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} + 3x^2 = 0$
   (e) $3y^2\dfrac{dy}{dx} + x\dfrac{dy}{dx} + x^3 = 0$
   (f) $y^3\dfrac{dy}{dx} + y + xy\dfrac{dy}{dx} + 3x^2 = 0$
   (g) $y^3\dfrac{dy}{dx} + xy + xy\dfrac{dy}{dx} + x^3 = 12$
   (h) $3y^2\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} + 3x^2 = 12$

   We differentiate implicitly on both sides, remembering to use the product rule on xy. Answer D.

11. Find the inverse function of $y = x^5 - 1$.

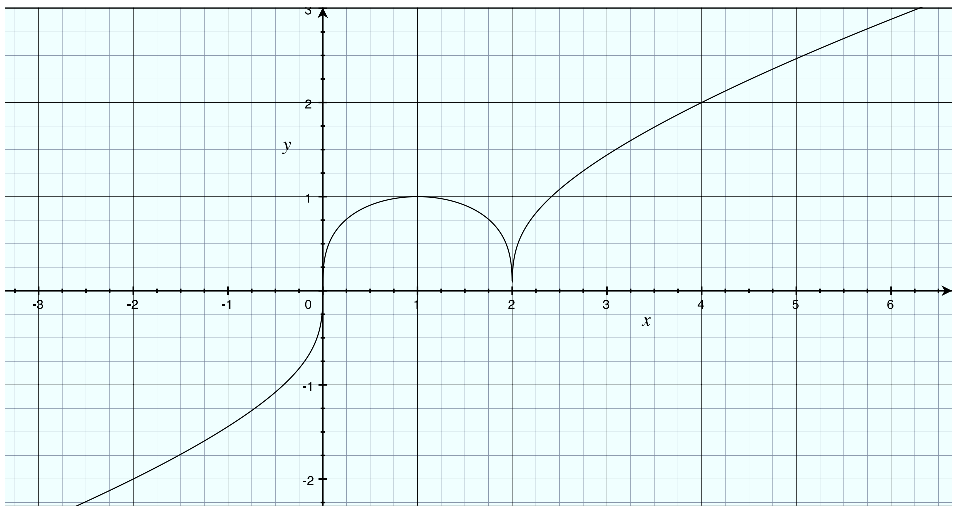
   (a) $\ln_5 x$
   (b) $\ln x$
   (c) $\sqrt[5]{x}$
   (d) $\sqrt[5]{x+1}$
   (e) $\sqrt[5]{x-1}$
   (f) $\sqrt[5]{(x-1)^2}$
   (g) $\sqrt[1]{x-5}$
   (h) $\sqrt[e]{x^5-1}$

We “switch and solve”:

$$\begin{aligned} x &= y^5 - 1 \\ x+1 &= y^5 \\ \sqrt[5]{x+1} &= y. \end{aligned}$$

Answer D.

12. For what values of x does the following graph <u>fail</u> to have a derivative?



(a) 0, 1, and 2 only.
(b) 0 and 1 only.
(c) 1 and 2 only.
(d) 0 and 2 only.
(e) 1 only.
(f) 0 only.
(g) 2 only.
(h) The function is differentiable everywhere.

Since the function has a vertical tangent at 0, and a cusp at 2, it is not differentiable at either of these points. (As you saw when you sketched this function on Worksheet 2!)
At $x = 1$, the function has a horizontal tangent, i.e., derivative 0; in particular is differentiable.

13. Which of the following are equivalent to the definition of $f'(0)$?

I. $\lim_{h\to 0} \dfrac{f(h)-f(0)}{h}$

II. $\lim_{x\to 0} \dfrac{f(x)-f(0)}{x}$

III. $\lim_{x\to 0} f(x) = f(0)$

(a) I, II, and III.
(b) I and II only.
(c) II and III only.
(d) I and III only.
(e) I only.
(f) II only.
(g) III only.
(h) None of the above.

I and II are the standard and alternative forms of the derivative, as discussed in class. III is the definition of continuity, and while a differentiable function is continuous, not every continuous function is differentiable; and in particular, the definitions are not equivalent. Answer B.

Problems 14-17: True/false. Each problem is worth 2 points.

14. True/false: if $f'(0) = 0$, then $\lim_{x\to 0} f(x) = f(0)$.

(a) True
(b) False

True! A differentiable function is continuous, by what we light-heartedly referred to as the "Chuck Norris Theorem".

15. True/false: if $f'(0) = 0$, then $f(0)$ must equal 0.

(a) True

(b) False

False. An example where this is not true is $f(x) = x^2 + 1$.

16. True/false: if $\lim_{x \to 0} f(x) = f(0)$, then f is differentiable at 0.

    (a) True
    (b) False

    False. A continuous function need not be differentiable: an example is $|x|$.

17. True/false: if $f(x)$ is any function (not necessarily continuous), and $f(-1) = -2$, $f(1) = 1$, then there is some c between -1 and 1 with $f(c) = 0$.

    (a) True
    (b) False

    False. An example where this is not true is

$$f(x) = \begin{cases} -2 & x < 0 \\ 1 & x \geq 0 \end{cases}.$$

Note that <u>if</u> we required f to be continuous, then the statement would be true by the IVT.

Part II (27 points): In each of the following problems, show your work clearly in the space provided. Partial credit will be given, and a correct answer without supporting work may not receive credit.

1. (5 points) Calculate $f'(x)$, where

$$f(x) = e^{\sin 2x} \cdot \tan x \cdot \sqrt{x^3 + \sin x^2}.$$

Explain your main steps. You don't need to simplify your answer.

$$\frac{d}{dx}f(x) = e^{\sin 2x} \cdot \cos 2x \cdot 2 + e$$

$$\frac{d}{dx}f(x) = e^{\sin 2x} \cdot \cos 2x \cdot 2(\tan x\sqrt{x^3+\sin x^2}) + e^{\sin 2x} \cdot \frac{d}{dx}\left[\tan x\sqrt{x^3+\sin x^2}\right]$$

(Chain, product rules)

$$= 2\cos 2x \cdot e^{\sin 2x} \cdot \tan x \cdot \sqrt{x^3+\sin x^2} + e^{\sin 2x}\sec^2 x\sqrt{x^3+\sin x^2} + e^{\sin 2x}\tan x\frac{d}{dx}\left[\sqrt{x^3+\sin x^2}\right]$$

(Product rule, remembering that $\frac{d}{dx}\tan x = \sec^2 x$)

$$= 2\cos 2x \cdot e^{\sin 2x} \cdot \tan x \cdot \sqrt{x^3+\sin x^2} + e^{\sin 2x}\sec^2 x\sqrt{x^3+\sin x^2} + e^{\sin 2x}\tan x\frac{3x^2+\cos x^2 \cdot 2}{2\sqrt{x^3+\sin x^2}}$$

(Chain rule applied twice to the $\sqrt{\ }$ and its "inside".)

2. Let $f(x) = \dfrac{1}{x^2-1}$.

   (a) (3 points) Find all horizontal asymptotes of $f(x)$. Explain why each is a horizontal asymptote.

   We recall a horizontal asymptote is when $\lim_{x\to\pm\infty} f(x)$ exists. We calculate:

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{x^2-1} &= 0\\ \lim_{x\to-\infty}\frac{1}{x^2-1} &= 0\end{aligned}$$

   so 0 is the only horizontal asymptote for f.

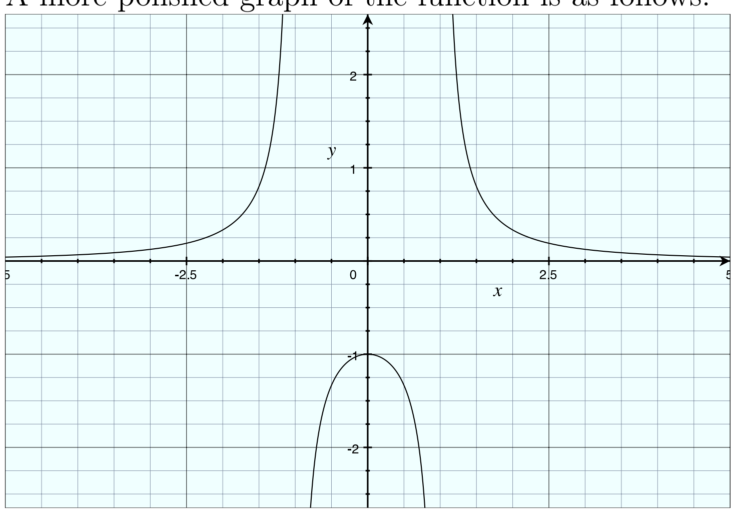
   (b) (4 points) Find all vertical asymptotes of $f(x)$. Explain why each is a vertical asymptote.

   We recall a horizontal asymptote is when $\lim_{x\to c\pm} f(x) = \infty$ or $-\infty$. The only possible vertical asymptotes are 1 and -1, since the function is continuous everywhere except those two points. We calculate:

$$\begin{aligned}
\lim_{x\to 1+} \frac{1}{x^2-1} &= \text{"}\frac{1}{\text{small }+}\text{"} = \infty \\
\lim_{x\to 1-} \frac{1}{x^2-1} &= \text{"}\frac{1}{\text{small -}}\text{"} = -\infty \\
\lim_{x\to -1+} \frac{1}{x^2-1} &= \text{"}\frac{1}{\text{small -}}\text{"} = \infty \\
\lim_{x\to -1-} \frac{1}{x^2-1} &= \text{"}\frac{1}{\text{small }+}\text{"} = \infty
\end{aligned}$$

(c) (4 points) Using your results of parts (a) and (b), sketch a rough graph of $f(x)$.

A more polished graph of the function is as follows:



Any result which shows the horizontal and vertical asymptotes clearly and following from the work on the previous parts received full credit.

3. (a) (3 points) State the limit definition of the derivatives $f'(x)$ and $g'(x)$.
Hint: It should start "$\lim_{h\to 0}\ldots$" or "$\lim_{z\to x}\ldots$."

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}
\end{aligned}
$$

(b) (8 points) Using the limit definition of the derivative, prove the sum rule.
To say the problem another way, use the limit definition to prove: if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

By definition,

$$\frac{d}{dx}[f(x) + g(x)] = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}.$$

We rearrange the right hand side to

$$\lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

and use a limit law (with the differentiability of f and g) to get

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

But from part (a), this is $f'(x) + g'(x)$. We have shown that

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x),$$

as desired. ■