Math 132
Midterm Examination 1 Solutions – February 15, 2012

6 multiple choice, 4 long answer. 100 points.

Part I was multiple choice. Only the correct answers are listed here.

1. Let $x_i = \dfrac{3i}{2n} - 1$ for $i = 0, 1, \dots, n$. These x_i's form a partition of the interval:

   (b) $[-1, \dfrac{1}{2}]$

2. Which of the following is equal to

$$\sum_{i=1}^{19} \frac{1}{i}$$

   (c) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{19}$.

3. Which of the following is an antiderivative of $\frac{1}{1+4x^2}$?

   (d) $\dfrac{1}{2} \tan^{-1}(2x)$

4. $\dfrac{d}{dx} \displaystyle\int_{-\pi}^{x} \sin t^2 \, dx$ is equal to

   This problem had an error, and was intended to be $\dfrac{d}{dx} \displaystyle\int_{-\pi}^{x} \sin t^2 \, dt$. As this is easy to miss in reading, we accepted the answer for the intended problem:

   (a) $\sin x^2$

   as well as:

   (i) None of the above.

5. If f is a continuous function such that $\int_0^{12} f(t)\,dt = 3$, $\int_2^{12} f(t)\,dt = 4$, and $\int_2^4 f(t)\,dt = 1$, then find $\int_0^4 f(t)\,dt$.

   (e) 0

   (Since $\int_0^4 f(t)\,dt = \int_0^{12} f(t)\,dt - \int_2^{12} f(t)\,dt + \int_2^4 f(t)\,dt$.)

6. Which of the following definite integrals have

$$\sum_{i=1}^{n}(1+\frac{i}{n})^4 \cdot \frac{2}{n}$$

as an associated Riemann sum?

I. $\int_0^2 (1+\frac{x}{2})^4\,dx$

II. $\int_1^3 x^4\,dx$

III. $\int_0^1 2\cdot(1+x)^4\,dx$

(f) I and III only.

Part II was long answer.

1. Integration

(a) (5 points) Evaluate $\displaystyle\int_{-1}^{1} \sin \pi x\,dx$.

Solutions 1: Substitute $u = \pi x$, so that $du = \pi\,dx$, and

$$\int_{-1}^{1} \sin \pi x\,dx = \int_{-\pi}^{\pi} \sin u\,\frac{1}{\pi}du = \left[-\cos u \cdot \frac{1}{\pi}\right]_{-\pi}^{\pi} = 0.$$

Solution 2: Since sin is an odd function, and $[-\pi,\pi]$ is symmetric around the y-axis, the integral is 0 by symmetry.

(b) (5 points) Evaluate $\displaystyle\int x\sin(x^2)\,dx$.

Substitute $u = x^2$, so that $du = 2x\,dx$, and

$$\int x\sin x^2\,dx = \int \sin u\,\frac{du}{2} = -\frac{\cos u}{2} + C = -\frac{\cos x^2}{2} + C.$$

(c) (5 points) Evaluate $\displaystyle\int_{\ln 2}^{\ln 3} e^{2x}\sqrt{1+e^x}\,dx$.

Substitute $u = 1+e^x$, so that $du = e^x\,dx$.

$$\begin{aligned}\int_{\ln 2}^{\ln 3} e^{2x}\sqrt{1+e^x}\,dx &= \int_3^4 (u-1)\cdot\sqrt{u}\,du = \int_3^4 u^{3/2} - u^{1/2}\,du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_3^4 \\ &= (\frac{2}{5}\cdot 4^{5/2} - \frac{2}{3}\cdot 4^{3/2}) - (\frac{2}{5}\cdot 3^{5/2} - \frac{2}{3}\cdot 3^{3/2}) \\ &= \frac{64}{5} - \frac{16}{3} - \frac{18\sqrt{3}}{5} + 2\sqrt{3}.\end{aligned}$$

(d) (7 points) Solve the initial value problem: If $\frac{d}{dt}f(t) = 8t \cdot (2t^2+1)^4$ and $f(0) = 1$, then find $f(t)$.

We first integrate $8t \cdot (2t^2+1)^4$. Substitute $u = 2t^2 + 1$, so that $du = 4t\,dt$, and we have

$$\int 8t \cdot (2t^2+1)^4\,dt = \int 2u^4\,du = 2\frac{u^5}{5} + C = 2 \cdot \frac{(2t^2+1)^5}{5} + C.$$

We now solve for C. At $t = 0$, we have

$$f(0) = 1 = 2 \cdot \frac{1^5}{5} + C = \frac{2}{5} + C,$$

so $C = \frac{3}{5}$, and $f(t) = 2 \cdot \frac{(2t^2+1)^5}{5} + \frac{3}{5}$.

2. Areas and volumes

(a) (15 points) Find the volume of the object formed by rotating

$$y = \sqrt{\frac{x}{1+x^2}}$$

about the x-axis for $0 \le x \le 2$.

The cross-sectional areas perpendicular to the x-axis have area $A(x) = \pi \cdot \frac{x}{1+x^2}$, so the volume is

$$V = \int_0^2 \frac{\pi x}{1+x^2}\,dx.$$

We substitute $u = 1 + x^2$ (so $du = 2x\,dx$) to get

$$= \int_1^5 \frac{\pi}{u}\frac{1}{2}du = \frac{\pi}{2}\left[\ln u\right]_1^5 = \frac{\pi}{2}\ln 5$$

(b) (10 points) Find the area between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$.

The curves cross when $\sin x = \cos x$, i.e., at $\frac{\pi}{4}$. Plotting test points at e.g. 0 and $\frac{\pi}{2}$, we determine that $\cos x > \sin x$ on $[0, \pi/4]$, and the reverse on $[\pi/4, \pi/2]$. The area is thus

$$\begin{aligned} &\left(\int_0^{\pi/4} \cos x - \sin x\,dx\right) + \left(\int_{\pi/4}^{\pi/2} \sin x - \cos x\,dx\right) \\ =\ & \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2} \\ =\ & \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1\right) + \left(-1 - 0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = 2\sqrt{2} - 2. \end{aligned}$$

(The area is symmetric over the line $x = \pi/4$, and this could be used to slightly shorten the calculations.)

3. The Fundamental Theorem of Calculus

(a) (6 points) Using definite integrals, give an antiderivative of $\sin x^2$. For full credit, explain clearly what theorems and/or properties of $\sin x^2$ that you are using.

By the FTC and continuity of $\sin x^2$,

$$\frac{d}{dx}\int_0^x \sin t^2\,dt = \sin x^2$$

(see also Problem 4). Hence,

$$\int_0^x \sin t^2\,dt$$

is an antiderivative for $\sin x^2$.

(b) (8 points) Find $\displaystyle\int_0^{\pi/2} (\frac{d}{dx} e^{x\sin x})\,dx$.

Since $e^{x\sin x}$ is an antiderivative for $\frac{d}{dx}e^{x\sin x}$, the integral is

$$= \left[e^{x\sin x}\right]_0^{\pi/2} = e^{\pi/2} - e^0.$$

4. Riemann sums and definite integrals

(a) (5 points) The points

$$1 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 3$$

form the uniform partition of $[1,3]$. Find x_i, and give the length of each part.

$$x_i = 1 + \frac{3-1}{n}\cdot i = 1 + \frac{2i}{n}; \qquad (x_i - x_{i-1}) = \frac{2}{n}.$$

(b) (6 points) Give any Riemann sum for $\displaystyle\int_1^3 \frac{1}{x}\,dx$. Be sure to explain the choice of partition and points that you make.

There are a number of possible answers to this problem; perhaps the easiest is to take the uniform partition $x_i = 1 + \frac{2i}{n}$ from part (a) with right endpoints (so that $c_i = x_i$), which gives

$$\sum_{i=1}^{n} f(x_i)\cdot(x_i - x_{i-1}) \quad = \quad \sum_{i=1}^{n} \frac{1}{1+\frac{2i}{n}}\cdot\frac{2}{n}.$$

(c) (4 points) In 1-3 sentences, explain why the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

from Worksheet 1 is not integrable on $[0, 1]$.

We showed on the worksheet that different rules for choosing the points c_i in the Riemann sums result in different limits. (Specifically, 0 and 1 from the "irrational rule" and "rational rule", respectively.) Since the definition of definite integral requires all choices of Riemann sums to converge to the same limit, the function is not integrable.