Ma 494 — Theoretical Statistics

Problem Set #3 — Due February 22, 2009

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1. Problem 5.5.6 page 398. (*Hints*: The density $f_Y(y,\theta)$ is gamma with parameters $\alpha = r$ equal to an integer and $\beta = \lambda = 1/\theta$. See Section 4.6.)

2. Problem 5.6.2 page 405.

(*Hint for part (b)*: Assume $S(X)$ for $X = (X_1, \ldots, X_n)$ is a sufficient statistic for θ and $L(\theta, X)$ is the likelihood. Then, if $S(X) = S(Y)$ for two samples X and $Y = (Y_1, \ldots, Y_n)$, the ratio $L(\theta, X)/L(\theta, Y)$ does not depend on θ. This can be used to show that particular statistics $S_0(X)$ are not sufficient.)

3. (Like Problem 5.6.8 page 405.) An *exponential family* is a density $f(x,\theta)$ that can be written in the form

$$f(x,\theta) = e^{K(x)p(\theta)+q(\theta)}A(x) \tag{1}$$

Show that $S(X_1, \ldots, X_n) = \sum_{i=1}^n K(X_i)$ is a sufficient statistic for θ.

4. (Like Problem 5.6.10 page 406.) Let $Y_1, \ldots, Y_n$ be an independent sample from the density $f(y,\theta) = \theta/(1+y)^{\theta+1}$ for $y > 0$ and $\theta > 0$.

(a) Find a sufficient statistic for θ.

(b) Show that $f(y,\theta)$ can be written as an exponential family as in (1) and find suitable functions $K(x)$, $p(\theta)$, $q(\theta)$, and $A(x)$.

5. Let $X_1, \ldots, X_n$ be an independent sample from $f(x,\theta)$ where

$$f(x,\theta) \;=\; (1/2)I_{(\theta-1,\theta+1)}(x) \;=\; \begin{cases} 1/2 & \text{if } \theta - 1 < x < \theta + 1 \\ 0 & \text{for other values of } x \end{cases} \tag{2}$$

Show that the vector-valued random variable $S(X) = (X_{\min}, X_{\max})$ is a sufficient statistic for θ. That is, $L(\theta, X_1, \ldots, X_n) = g(X_{\min}, X_{\max}, \theta)h(X_1, \ldots, X_n)$ for functions $g(y_1, y_2, \theta)$ and $h(x)$, and find suitable functions g and h.

Remarks: If a likelihood $L(\theta, X)$ is maximized on an interval, then any statistic $S(X)$ with values in that interval is a maximum likelihood estimator. Using this, one can show that $S(X) = (X_{\min} + X_{\max})/2$ is a maximum likelihood estimator of θ.

(*Warning:* Both $f(x,\theta)$ and $L(\theta, X)$ take on only the values 0 and 1, so that setting $(\partial/\partial\theta)\log L(\theta, X) = 0$ and solving for θ may not be helpful. Either work with inequalities or with indicator functions $I_{(\text{set})}(x)$ as in equation (2). Note that $I_{(\theta-1,\theta+1)}(x) = I_{(x-1,x+1)}(\theta)$, since both are equal to one if and only if $|x - \theta| < 1$.)