Ma 494 — Theoretical Statistics

Solutions for Problem Set #5 — Due March 24, 2010

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NOTE: 5 problems on 4 pages.

1. (i) $\widehat{p} = M/N = 0.94$ for $N = 1000$ simulated samples of size n. By Theorem 5.3.1 on page 369 of the text (with $X = M$ and $n = N$), a symmetric 95% confidence interval for the true coverage probability p is

$$\left(\widehat{p} - 1.960\sqrt{\frac{\widehat{p}(1-\widehat{p})}{1000}},\ \widehat{p} + 1.960\sqrt{\frac{\widehat{p}(1-\widehat{p})}{1000}}\right) = (0.94 - 0.015,\ 0.94 + 0.015) = (0.925, 0.955)$$

This contains $p = 0.95$, so that $M/N = 0.94$ is consistent with $p = 0.95$.

(ii) If $N = 1000$ is replaced by $N = 10{,}000$, the confidence interval is

$$\left(\widehat{p} - 1.960\sqrt{\frac{\widehat{p}(1-\widehat{p})}{10{,}000}},\ \widehat{p} + 1.960\sqrt{\frac{\widehat{p}(1-\widehat{p})}{10{,}000}}\right) = (0.94 - 0.005,\ 0.94 + 0.005) = (0.935, 0.945)$$

This DOES NOT contain $p = 0.95$, so that $M/N = 0.94$ is NOT consistent with $p = 0.95$.

2. Since Y has a gamma distribution with parameters (r, θ) (see Definition 4.6.2 on page 329 in the text), the density of Y given θ is

$$f_Y(y, \theta) = \frac{\theta^r}{\Gamma(r)} y^{r-1} e^{-\theta y}$$

Note that r is assumed fixed. The prior distribution on θ is gamma with parameters (s, μ), so that the prior density (for θ) is

$$\pi_0(\theta) = \frac{\mu^s}{\Gamma(s)} \theta^{r-1} e^{-\mu\theta}$$

Thus the Bayesian joint probability density of (θ, y) is

$$f_{\Theta,Y}(\theta, y) = \pi_0(\theta) f_Y(y, \theta) = \frac{\mu^s}{\Gamma(s)} \theta^{s-1} e^{-\mu\theta} \frac{\theta^r}{\Gamma(r)} y^{r-1} e^{-\theta y}$$

$$= \frac{\mu^s y^{r-1}}{\Gamma(s)\Gamma(r)}\, \theta^{r+s-1} e^{-\theta(\mu+y)} = C(y)\, \theta^{r+s-1} e^{-\theta(\mu+y)} \qquad (1)$$

(i) By definition, the Bayesian posterior distribution given $y = Y$ is the two-dimensional density in (1) conditional on $y = Y$. By Definition 3.11.1 page 250 in the text, this conditional density is of the same form

$$\pi_1(\theta \mid Y) \;=\; f_{\Theta|Y}(\theta \mid Y) \;=\; C_1(Y)\,\theta^{r+s-1}e^{-\theta(\mu+Y)} \tag{2}$$

where $\int \pi_1(\theta \mid Y)\,d\theta = 1$. The form of (2) as a function of θ indicates that $\pi_1(\theta \mid Y)$ is a gamma density with parameters $(r+s, \mu+Y)$ (see Definition 4.6.2 page 329). Since $\int \pi_1(\theta \mid Y)\,d\theta = 1$, it follows that $C_1(Y) = (\mu+Y)^{r+s}/\Gamma(r+s)$.

(ii) By Theorem 5.8.1 page 420 in the text, the Bayes estimator $\widehat{\theta}_B$ of θ for one observation Y with square-loss risk is the same as the mean of the posterior density $\pi(\theta \mid Y)$. By Theorem 4.6.3 page 330 in the text, if W has a gamma distribution with parameters (t, λ) then $E(W) = t/\lambda$ and $\mathrm{Var}(W) = t/\lambda^2$. By (2), the distribution of Θ given Y is gamma with parameters $(r+s, \mu+Y)$. Thus $\widehat{\theta}_B = \int \theta\pi_1(\theta \mid Y)\,d\theta = E(\Theta \mid Y) = t/\lambda = (r+s)/(\mu+Y)$.

3. (i) The likelihood is

$$L(p, Y_1, \ldots, Y_n) \;=\; \prod_{j=1}^{n}(1-p)^{Y_j-1}p \;=\; (1-p)^{\sum_{j=1}^{n}(Y_j-1)}p^n$$

Thus if $Y = (Y_1, \ldots, Y_n)$

$$\log L(p, Y) \;=\; \left(\left(\sum_{j=1}^{n} Y_j\right) - n\right)\log(1-p) + n\log(p)$$

$$\frac{\partial}{\partial p}\log L(p, Y) \;=\; \frac{n}{p} \;-\; \frac{\left(\sum_{j=1}^{n} Y_j\right) - n}{1-p} \tag{3}$$

Setting the expression in (3) equal to zero leads to

$$p\left(\left(\sum_{j=1}^{n} Y_j\right) - n\right) \;=\; n(1-p) \;=\; p\left(\sum_{j=1}^{n} Y_j\right) - np \;=\; n - np$$

Thus the maximum likelihood estimator is

$$\widehat{p} \;=\; \frac{n}{\sum_{j=1}^{n} Y_j} \;=\; \frac{1}{\overline{Y}} \tag{4}$$

where $\overline{Y} = (1/n)\sum_{j=1}^{n} Y_j$. Since $Y_j \ge 1$ for all j, $\overline{Y} \ge 1$ and $0 \le \widehat{p} \le 1$, as expected.

(ii) Since $f(y,p) = (1-p)^{y-1}p$ for one observation, it follows from (3) with $n = 1$ that

$$\frac{\partial}{\partial p}\log f(Y,p) \;=\; \frac{1}{p} \;-\; \frac{Y-1}{1-p}, \qquad -\frac{\partial^2}{\partial p^2}\log f(Y,p) \;=\; \frac{1}{p^2} \;+\; \frac{Y-1}{(1-p)^2} \tag{5}$$

By Theorem 4.4.1 on page 318 of the text, $E(Y) = 1/p$. (This also follows from (5) and $E\big((\partial/\partial p)\log f(Y,p)\big) = 0$.) Thus by (5), the Fisher information is

$$\begin{aligned} I(f,p) \;&=\; -E\left(\frac{\partial^2}{\partial p^2}\log f(Y,p)\right) \;=\; \frac{1}{p^2} \;+\; \frac{E(Y)-1}{(1-p)^2} \\ &=\; \frac{1}{p^2} \;+\; \frac{1-p}{p(1-p)^2} \;=\; \frac{1}{p}\left(\frac{1}{p} \;+\; \frac{1}{1-p}\right) \;=\; \frac{1}{p^2(1-p)} \end{aligned} \tag{6}$$

By equation (5.6) in the Math 494 notes,

$$\left(\widehat{p} - \frac{1.96}{\sqrt{nI(f,\widehat{p})}}\,,\;\; \widehat{p} + \frac{1.96}{\sqrt{nI(f,\widehat{p})}}\right)$$

is an asymptotic central 95% confidence interval for p. By (4) and (6), $\widehat{p} = 1/\overline{Y}$ and

$$\frac{1}{nI(f,\widehat{p})} \;=\; \frac{\widehat{p}^2(1-\widehat{p})}{n} \;=\; \frac{\overline{Y}-1}{n\overline{Y}^3}$$

By assumption, $\overline{Y} = 3$ and $n = 100$. Thus the asymptotic 95% confidence interval for p is

$$\left(\frac{1}{\overline{Y}} \;-\; 1.960\sqrt{\frac{\overline{Y}-1}{n\overline{Y}^3}}\,,\;\; \frac{1}{\overline{Y}} \;+\; 1.960\sqrt{\frac{\overline{Y}-1}{n\overline{Y}^3}}\right) \;=\; (0.2800,\ 3866)$$

4. Under hypothesis H_0, the Y_i are normally distributed $N(95, 15^2)$ so that $\overline{Y}$ is $N(95, 15^2/22)$. If λ satisfies $\alpha = P(\overline{Y} \ge \lambda) = 0.06$, then

$$P\left(\frac{\overline{Y}-95}{\sqrt{15^2/22}} \;\ge\; \frac{\lambda-95}{\sqrt{15^2/22}}\right) \;=\; P\left(Z \ge \frac{\lambda-95}{\sqrt{15^2/22}}\right) \;=\; 0.06$$

where Z is standard normal ($N(0,1)$). This implies $(\lambda - 95)/\sqrt{15^2/22} = 1.555$ by Table A.1 and

$$\lambda \;=\; 1.555\sqrt{\frac{15^2}{22}} \;+\; 95 \;=\; 99.97$$

Thus, values of $\overline{Y} > 99.97$ cause H_0 to be rejected using the standard one-sided test at level of significance $\alpha = 0.06$.

5. (i) Here

$$\alpha \;=\; P(X \geq 0.90 \mid H_0) \;=\; \int_{0.90}^{1} f(x,1)\,dx \;=\; \int_{0.90}^{1} 2x\,dx$$
$$=\; x^2\Big]_{x=0.90}^{x=1} \;=\; 1-0.90^2 \;=\; 1-0.81 \;=\; 0.19$$

(ii) If $H_1 : \theta = 10$, the power is

$$\text{Pow} \;=\; P(X \geq 0.90 \mid H_1) \;=\; \int_{0.90}^{1} f(x,10)\,dx \;=\; \int_{0.90}^{1} 10x^{11}\,dx$$
$$=\; x^{11}\Big]_{x=0.90}^{x=1} \;=\; 1-0.90^{11} \;=\; 1-0.314 \;=\; 0.686$$

and $\beta = 1 - \text{Pow} = 1 - 0.686 = 0.314$.