Ma 494 — Theoretical Statistics

Test #1 — February 19, 2009

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Closed book and closed notes. One $8\frac{1}{2} \times 11$ sheet of paper with notes on both sides and a calculator are allowed.

1. Let $f(x,r) = r\exp(-r/x)(1/x^2)$ for $x > 0$, $r > 0$. Given an independent sample $X_1, \ldots, X_n$ from $f(x,r)$, find the maximum likelihood estimator of r.

2. Let $f(x,\theta) = \frac{1}{2}e^{-|x-\theta|}$ for $-\infty < x < \infty$, $-\infty < \theta < \infty$. Let $X_1, \ldots, X_n$ be an independent sample from $f(x,\theta)$. Prove that the sample mean $\overline{X} = (1/n)\sum_{k=1}^{n} X_k$ is an unbiased estimator of θ.

3. Let $f(x,\mu) = e^{-\mu}\mu^x/x!$ for $x = 0, 1, 2, \ldots$ be the Poisson distribution. Recall that $E(X) = \mathrm{Var}(X) = \mu$ if X has this distribution. Let $X_1, \ldots, X_n$ be an independent sample from $f(x,\mu)$.

(i) Find the scores $Y_k(\mu)$ of X_k and the Fisher information of X_k.

(ii) Prove that the sample mean $\overline{X}$ is an efficient estimator of μ (that is, its variance attains the Cramér-Rao lower bound).

(*Hint*: You can do parts (i,ii) in any order.)

4. Let $f(x,p) = (x+1)p^x(1-p)^2$ for integers $x = 0, 1, 2, \ldots$ and $0 < p < 1$. Given an independent sample $X_1, \ldots, X_n$ from $f(x,p)$, find the maximum likelihood estimator of p.

5. Let $f(x,\theta) = 4\theta x^3 e^{-\theta x^4}$ for $x > 0$ and $\theta > 0$. Given an independent sample $X_1, \ldots, X_n$ from $f(x,\theta)$, find a sufficient statistic for θ (and show that it is sufficient).