Links in the complex of separable graphs

John Shareshian
California Institute of Technology
Pasadena, CA 91125
email: shareshi@cco.caltech.edu

September 12, 2000

Abstract

We determine the homotopy type of the link of a nonempty face in the complex of not 2-connected graphs on n vertices. Corollaries of our main result are that the complex of not 2-connected 3-uniform hypergraphs on n vertices has the homotopy type of a wedge of spheres of dimension $n-4$ and that the lattice of block-closed graphs on n vertices is Cohen-Macaulay.

1 Introduction and statement of results

For $n \in \mathbf{N}$, let $\Delta(n)$ be the simplex with one vertex for each edge in the complete graph K_n, so the vertices of $\Delta(n)$ are indexed by the set $\binom{[n]}{2}$ of 2-subsets of $[n]$. The faces of $\Delta(n)$ correspond to graphs on vertex set $[n]$ in the natural way, with an m-dimensional face corresponding to a graph with $m+1$ edges. In what follows, no distinction will be made between a graph G and the face of $\Delta(n)$ corresponding to G.

The basic notations and definitions from topology which appear here without explanation are explained in [Bj]. A graph G on vertex set $V = V(G)$ with edge set $E = E(G) \subseteq \binom{V}{2}$ will sometimes be denoted by (V, E). Recall that $G = (V, E)$ is called *2-connected* if G is connected and for every $v \in V$ the subgraph of G induced on $V \setminus \{v\}$ is connected. From now on, we will write G_W for the subgraph of G induced on any $W \subseteq V$. We will call G *separable* if G is not 2-connected. If (V, E) is separable and $F \subseteq E$ then (V, F) is also separable. Thus the set

$$\Delta_n^2 := \{G \in \Delta(n) : G \text{ is separable}\}$$

is a subcomplex of $\Delta(n)$. The topology of Δ_n^2 is important in Vassiliev's study of knot invariants (see [V1, V2]). The following result was proved independently in [BBLSW] and [T1].

Theorem 1.1 *The complex Δ_n^2 has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $2n-5$.*

Further results on the homology of Δ_n^2 (actually that of the quotient complex $\Delta(n)/\Delta_n^2$) appear in [T2] and [Sh]. Here we will study the topology of the links $lk_{\Delta_n^2}(G)$ of the faces of Δ_n^2, in order to understand the topology of another complex arising in Vassiliev's work, which is described below.

Recall that a *block* of a graph $G=(V,E)$ is a subset W of V such that G_W is 2-connected and for every proper superset X of W in V, G_X is separable. Also, a *cutpoint* of G is a vertex $v \in V$ such that $G_{V\setminus\{v\}}$ has more connected components than G. The following result is well known (see for example [Lo]).

Proposition 1.2 *Let $G=(V,E)$ be a graph. There is a unique decomposition $V=\cup_{i=1}^{r} W_i$ such that each W_i is a block of G. If $1 \leq i < j \leq r$ then $|W_i \cap W_j| \leq 1$. All edges of G connect two points in the same block. The cutpoints of G are precisely those vertices which are contained in more than one block. Let $B(G)$ be the bipartite graph with one vertex for each cutpoint c of G and one vertex for each block W of G, with an edge $\{c,W\}$ if and only if $c \in W$. Then each connected component of $B(G)$ is a tree. In particular, if G is connected then $B(G)$ is a tree.*

We call a graph G *block-closed* if for each block W of G, the induced subgraph G_W is a clique. For any graph G, let $c(G)$ be the number of connected components of G and let $b(G)$ be the number of blocks of G which contain at least two vertices. Our main theorem is as follows.

Theorem 1.3 *Let $H \in \Delta_n^2$. Then*

(a) If H is not block-closed then $lk_{\Delta_n^2}(H)$ is contractible, and

(b) If H is block closed then $lk_{\Delta_n^2}(H)$ has the homotopy type of a wedge of spheres of dimension $2c(G)+b(G)-5$.

The main reason for proving Theorem 1.3 is to understand the topology of another class of complexes which arises in Vassiliev's studies of invariants of maps of the circle into $\mathbf{R}^3$ whose image contains no threefold self-intersections and invariants of ornaments (see [V3]). Recall that a *hypergraph* $\mathcal{H}$ on vertex set $[n]$ is simply a collection of subsets of $[n]$. For $k \in [n]$, $\mathcal{H}$ is called *k-uniform* if each $X \in \mathcal{H}$ satisfies $|X| = k$. In particular, a 2-uniform hypergraph is a graph. The *underlying graph* of a hypergraph $\mathcal{H}$ on $[n]$ is the graph $G_{\mathcal{H}}$ on $[n]$ with $\{i,j\} \in E(G_{\mathcal{H}})$ if and only if there is some $X \in \mathcal{H}$ with $\{i,j\} \subseteq X$. For $n \in \mathbf{N}$ and $k \in [n]$, let $\Delta(n,k)$ be the simplex with one vertex for each k-subset of $[n]$. Then, as with the case $k=2$, $\Delta(n,k)$ has one face for each k-uniform hypergraph on $[n]$, and the set

$$\Delta_{n,k}^2 := \{\mathcal{H} \in \Delta(n,k) : G_{\mathcal{H}} \text{ is separable}\}$$

is a subcomplex of $\Delta(n,k)$. As noted above in the cases $k=2,3$, the complexes $\Delta^2_{n,k}$ for arbitrary n,k (actually, the quotient complexes $\Delta(n,k)/\Delta^2_{n,k}$) arise in work of Vassiliev. Some results on these complexes appear in [BBLSW]. Combining Theorem 1.3 with some of the results in [BBLSW] gives the following corollary (see Conjecture 9.5 of [BBLSW]).

Corollary 1.4 *For $n \geq 3$, the complex $\Delta^2_{n,3}$ has the homotopy type of a wedge of spheres of dimension $n-4$. If a_n is the number of spheres in this wedge, then the generating function $P(x) := \sum_{n\geq 3}(-1)^n a_n \frac{x^n}{n!}$ satisfies*

$$P'(x) = \ln\left(\frac{-x(x-2)}{(x-1)+\sqrt{2-(x-1)^2}}\right).$$

The proof of Corollary 1.4 involves examining the poset $\Sigma_{n,2}$ of block-closed graphs on $[n]$, where for block-closed graphs $G=([n],E(G))$ and $H=([n],E(H))$ we have $G \leq H$ if and only if $E(G) \subseteq E(H)$. Note that $\Sigma_{n,2}$ has a unique minimum element (the graph $\emptyset$ with no edges) and a unique maximum element (the complete graph K_n, which is the only 2-connected block-closed graph). In fact, as noted in [BBLSW], $\Sigma_{n,2}$ is a lattice.

Recall that for a poset P, the order complex $\Delta(P)$ is the simplicial complex whose vertices are the elements of P and whose simplices are the subsets of P which are totally ordered by the relation defining P. For $x,y \in P$, we write (x,y) for the subposet of P consisting of all $z \in P$ with $x < z < y$, and we write $\Delta(x,y)$ for the order complex of this subposet. The following result follows easily from Quillen's fiber lemma. A proof appears in the next section.

Proposition 1.5 *Let $H \in \Delta^2_n$. Then $lk_{\Delta^2_n}(H)$ is homotopy equivalent to the order complex $\Delta(P(H))$, where $P(H)$ is the subposet of $\Sigma_{n,2} \setminus \{K_n\}$ consisting of all block-closed graphs which contain H as a proper subgraph.*

Theorem 1.3 and Proposition 1.5 can be combined with results from [BBLSW] to give the following result (see Section 11 of [Bj] for definitions, basic results and references on Cohen-Macaulay posets and complexes).

Corollary 1.6 *For $n \geq 2$, the poset $\Sigma_{n,2}$ is Cohen-Macaulay.*

2 The proofs

We now provide the proofs of Proposition 1.5, Theorem 1.3, and Corollaries 1.4 and 1.6, in the order just listed. Before doing so, we introduce some additional notation. For graphs G,K on the same vertex set V, $G \cup K$ will denote the graph on V with edge set $E(G) \cup E(K)$. The graphs $G \cap K$ and $G \setminus K$ are defined similarly. Also, we write $G \subseteq K$ if $E(G) \subseteq E(K)$ (that is, if G is a subgraph of K). For an edge $e = \{x, y\} \in E(G)$, $G - e$ or $G - \{x, y\}$ will denote the graph obtained from G by removing e from $E(G)$. If $e = \{x, y\} \notin E(G)$ then $G + e$ or $G + \{x, y\}$ will denote the graph obtained from G by adding e to $E(G)$.

Fix $H \in \Delta_n^2$. Let $\mathcal{FP}(H)$ be the poset whose elements are the nonempty simplices in $lk_{\Delta_n^2}(H)$, ordered by inclusion. Then $lk_{\Delta_n^2}(H)$ is homeomorphic to $\Delta(\mathcal{FP}(H))$ (see Section 9 of [Bj]), so to prove Proposition 1.5 it suffices to show that $\Delta(\mathcal{FP}(H))$ is homotopy equivalent to $\Delta(P(H))$. For $G \in \mathcal{FP}(H)$ let $\overline{G}$ be the unique block-closed graph with the same blocks as $G \cup H$. Define $f : \mathcal{FP}(H) \to P(H)$ by $G \mapsto \overline{G}$. Then f is a well-defined, order preserving map. For $K \in P(H)$, set $f_{\leq K} := \{G \in \mathcal{FP}(H) : f(G) \leq K\}$. Then $f_{\leq K}$ is the subposet of $\mathcal{FP}(H)$ consisting of all G such that $G \subseteq K \setminus H$. Since $f_{\leq K}$ has a unique maximum element $K \setminus H$, $\Delta(f_{\leq K})$ is a cone whose apex is the vertex corresponding to $K \setminus H$, and is therefore contractible. Proposition 1.5 now follows from Quillen's fiber lemma (see Section 1 of [Qu] or Theorem 10.5 of [Bj]).

Theorem 1.3(a) now follows easily, since if H is not block-closed then the block-closed graph whose blocks are those of H is the unique minimum element of $P(H)$, and therefore $\Delta(P(H))$ is a (contractible) cone.

To prove part Theorem 1.3(b), we first note that the claim therein holds for small n (say $n = 2, 3$) by observation, so we can proceed by induction on n. We will need a collapsibility result which is a special case of Forman's discrete Morse theory (see [Fo], also see [Ch, Sh] for a description of the theory in the language that will be used below).

For a simplicial complex Σ, define $D(\Sigma)$ to be the directed graph whose vertices are (indexed by) the faces of Σ (including the empty face), with an arc $\sigma \to \tau$ if and only if τ is a maximal face of σ in Σ. That is, $D(\Sigma)$ is obtained from the Hasse diagram of the face poset of Σ by directing each edge downwards. For a set X of arcs in $D(\Sigma)$, define $D_X(\Sigma)$ to be the directed graph obtained by reversing the direction of all the arcs in X. Thus if $A(\Sigma)$ is the set of arcs in $D(\Sigma)$ and $A_X(\Sigma)$ is the set of arcs in $D_X(\Sigma)$ then $A_X(\Sigma) = (A(\Sigma) \setminus X) \cup \{\tau \to \sigma : \sigma \to \tau \in X\}$.

Recall that a *perfect matching* in a directed graph D is a set M of arcs such that each vertex of D is an endpoint (at either end) of exactly one arc in M, and that D is *acyclic* if there is no directed cycle in D. We will call a set M of arcs in $D(\Sigma)$ an *acyclic perfect matching* if M is a perfect matching in $D(\Sigma)$ and $D_M(\Sigma)$ is acyclic. The next proposition, which is a special case of Theorem 3.5 of [Fo], is easy to prove by induction on the number of faces of Σ (see Proposition 3.7 of [BBLSW]).

Proposition 2.1 *Let Σ be a simplicial complex. If $D(\Sigma)$ contains an acyclic perfect matching then Σ is collapsible.*

See Section 11 of [Bj] for facts about collapsibility, the most important of which for our purposes is that a collapsible complex is contractible. Proposition

2.1 will be used in conjunction with the following proposition (see [BW] or Lemma 10.2 of [Bj]) to produce a quotient complex of $lk_{\Delta_n^2}(H)$ which has the same homotopy type but a more transparent structure.

Proposition 2.2 *Let Σ be a simplicial complex and let Γ be a contractible subcomplex of Σ. Then Σ is homotopy equivalent to the quotient complex Σ/Γ.*

See [LW] for basic facts on quotient complexes, along with facts about cell complexes which will be used below.

Fix a block-closed graph $H \in \Delta_n^2$. Note that if $E(H) = \emptyset$ then $lk_{\Delta_n^2}(H) = \Delta_n^2$. In this case we have $c(G) = n$, $b(G) = 0$, and Theorem 1.3(b) is part of Theorem 1.1. We now assume that $E(H) \neq \emptyset$. Since H is separable, $H \neq K_n$. Since H is neither empty nor complete and $lk_{\Delta_n^2}(K)$ is homeomorphic with $lk_{\Delta_n^2}(H)$ for any graph K which is isomorphic to H, we may (and do) make the following assumptions:

- The edge $\{1, n\}$ is not in $E(H)$, and
- For some $1 < i < n$ the edge $\{i, n\}$ is in $E(H)$.

Let $\mathcal{I}_1$ be the star of $\{1, n\}$ in $lk_{\Delta_n^2}(H)$ (see 9.9 of [Bj]). Note that $lk_{\Delta_n^2}(H) \setminus \mathcal{I}_1$ consists of those $G \in lk_{\Delta_n^2}(H)$ such that $\{1, n\} \notin E(G)$ and $(G \cup H) + \{1, n\}$ is 2-connected.

Lemma 2.3 *Let K be a separable graph with $|V(K)| > 2$. Assume $\{x, y\} \notin E(K)$ and $K + \{x, y\}$ is 2-connected. Then K has blocks $W_1, W_2, \ldots, W_r$ with $x \in W_1$, $y \in W_r$ and $W_i \cap W_j \neq \emptyset$ if and only if $|i - j| \leq 1$. In other words, $B(K)$ is a path with endpoints W_1 and W_r.*

Proof: First note that if K is not connected then (as $|V(K)| > 2$) either $K + \{x, y\}$ is not connected or one of x, y is a cutpoint of $K + \{x, y\}$. Both of these conditions are impossible. Therefore, $B(K)$ is a tree. Note that each vertex of degree one in $B(K)$ corresponds to a block of K. If K does not have the structure described in the lemma, then there is some vertex of degree one corresponding to a block which contains neither x nor y. Let c be the unique cutpoint of K contained in that block. Then c is a cutpoint of $K + \{x, y\}$, which is impossible. $\Box$

For $G \in lk_{\Delta_n^2}(H) \setminus \mathcal{I}_1$ let $m(G)$ be the unique cutpoint of $G \cup H$ contained in the block of $G \cup H$ which contains vertex 1. Note that Lemma 2.3 guarantees the existence of $m(G)$. Define

$$\mathcal{I}_2 := \left\{ G \in lk_{\Delta_n^2}(H) \setminus \mathcal{I}_1 : \{m(G), n\} \notin E(H) \right\},$$

and set

$$\mathcal{I} := \mathcal{I}_1 \cup \mathcal{I}_2.$$

By definition, $\mathcal{I}_1$ is a subcomplex of $lk_{\Delta_n^2}(H)$. If $G \in \mathcal{I}_2$ and $e \in E(G)$ then either $G - e \in \mathcal{I}_1$ or $m(G-e)$ is contained in the block of $G \cup H$ which contains 1. In the second case we have $\{m(G-e), n\} \notin E(H)$, so $G - e \in \mathcal{I}_2$. Therefore, $\mathcal{I}$ is a subcomplex of $lk_{\Delta_n^2}(H)$.

Lemma 2.4 *The directed graph $D(\mathcal{I})$ contains an acyclic perfect matching.*

Proof: Let $X = \{G \in \mathcal{I}_1 : \{1,n\} \in E(G)\}$ and let M_1 be the set of arcs

$$M_1 := \{G \to G - \{1,n\} : G \in X\}.$$

Let $Y = \{G \in \mathcal{I}_2 : \{m(G), n\} \in E(G)\}$ and set

$$M_2 := \{G \to G - \{m(G), n\} : G \in Y\}.$$

Set $M = M_1 \cup M_2$. We will see that M is an acyclic perfect matching in $D(\mathcal{I})$.

It follows immediately from the definitions that M_1 is a perfect matching in $D(\mathcal{I}_1)$, so to show that M is a perfect matching in $D(\mathcal{I})$ it suffices to show that both endpoints of each arc in M_2 lie in $\mathcal{I}_2$ and that each element of $\mathcal{I}_2$ is the endpoint of a unique arc in M_2. Say $G \in Y$. Then $G \cup H$ has exactly two blocks W_1, W_2 with $1 \in W_1$, $n \in W_2$, and $m(G)$ is the unique cutpoint of $G \cup H$. Let K be the subgraph of $G \cup H$ induced on W_2, so K is 2-connected. If $K - \{m(G), n\}$ is 2-connected then the blocks of $(G \cup H) - \{m(G), n\}$ are W_1 and W_2, and $m(G - \{m(G), n\}) = m(G)$, so $G - \{m(G), n\} \in \mathcal{I}_2$.

Say $K - \{m(G), n\}$ is separable. If $|W_2| = 2$ then $\{m(G), n\}$ is the only edge in $E(G \cup H)$ containing n. This contradicts our assumption that there is some $\{i, n\}$ in $E(H)$, so we must have $|W_2| > 2$. By Lemma 2.3, the blocks of $K - \{m(G), n\}$ are $V_1, \ldots, V_r$ with $m(G) \in V_1$, $n \in V_r$ and $V_i \cap V_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Now the blocks of $(G \cup H) - \{m(G), n\}$ are $W_1, V_1, \ldots, V_r$ and $m(G - \{m(G), n\}) = m(G)$. It follows that $G - \{m(G), n\} \in \mathcal{I}_2$. Therefore, both endpoints of any arc in M_2 lie in $\mathcal{I}_2$.

If $G \in Y$ then G is an endpoint of the arc $G \to G - \{m(G), n\} \in M_2$, and any other arc in M_2 which has G as an endpoint is of the form $K \to G$ with $K \in Y$. Assume for contradiction that such an arc exists. Since $G \in Y$, we know that $G \cup H$ has exactly two blocks. Since $K \cup H$ is separable, the blocks of $K \cup H$ are those of $G \cup H$ and $m(K) = m(G)$. The desired contradiction now follows from the fact that $\{m(K), n\} \in E(G)$. Therefore, G is the endpoint of a unique arc in M_2.

If $G \in \mathcal{I}_2 \setminus Y$ then any arc in M_2 which has G as an endpoint is of the form $K \to G$ with $K \in Y$, so $G = K - \{m(K), n\}$. Since vertices 1 and n are in distinct blocks of any graph in $\mathcal{I}_2$, we see that if such a K exists then $m(K) = m(G)$ and $K = G + \{m(G), n\}$. Now note that $m(G)$ is the unique cutpoint of $(G \cup H) + \{m(G), n\}$. It follows that $G + \{m(G), n\} \in Y$, so the arc $G + \{m(G), n\} \to G$ is the unique arc in M_2 having G as an endpoint. Therefore, M is a perfect matching in $D(\mathcal{I})$.

Since $\mathcal{I}_1$ is a subcomplex of $\mathcal{I}$ and no arc in M has one endpoint in $\mathcal{I}_1$ and the other in $\mathcal{I}_2$, there is no arc in $D_M(\mathcal{I})$ of the form $G \to K$ with $G \in \mathcal{I}_1$ and $K \in \mathcal{I}_2$. It follows that any directed cycle in $D_M(\mathcal{I})$ either has all vertices in $\mathcal{I}_1$ or has all vertices in $\mathcal{I}_2$. Now note that any directed cycle in $D_M(\mathcal{I})$ is of the form

$$A_1 \to B_1 \to A_2 \to B_2 \to \ldots \to A_r \to B_r \to A_{r+1} = A_1,$$

where there are edges $a_1, \ldots, a_r, b_1, \ldots, b_r$ with $B_i = A_i + a_i$ and $A_{i+1} = B_i - b_i$ for all $i \in [r]$. Also, the multisets $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ are equal. To see that these claims hold, let

$$G_1 \to G_2 \to \ldots \to G_s = G_1$$

be any directed cycle in $D_M(\mathcal{I})$. For each $i \in [s-1]$, either G_i is a maximal proper subgraph of G_{i+1} or G_{i+1} is a maximal proper subgraph of G_i. Since M is a matching, it cannot be the case that $|E(G_i)| < |E(G_{i+1})| < |E(G_{i+2})|$ for $i \in [s-2]$ or that $|E(G_{s-1})| < |E(G_1)| < |E(G_2)|$. The first claim follows. Since $A_{r+1} = A_1$ is obtained from A_1 by adding all the a_i and removing all the b_i, the second claim also holds.

Now assume for contradiction that there is a cycle of the form just described in $D_M(\mathcal{I})$ with all the A_i and all the B_i contained in $\mathcal{I}_1$. Then $B_i \to A_i \in M_1$ for each i, and it follows that each $a_i = \{1, n\}$. Therefore, each $b_i = \{1, n\}$ and $A_2 = A_1$, giving the desired contradiction.

Finally, assume for contradiction that there is a cycle of the given form in $D_M(\mathcal{I})$ with all the A_i and all the B_i contained in $\mathcal{I}_2$. Then each $a_i = \{m(B_i), n\}$, so n is an endpoint of each b_i. Since n is not in the block of $B_i \cup H$ containing 1, the block of $A_{i+1} \cup H$ containing 1 is the same as the block of $B_i \cup H$ containing 1, so $m(A_{i+1}) = m(B_i)$ for each $i \in [r]$. By the same argument, $m(A_i) = m(B_i)$ for each i. Therefore, $a_i = \{m(A_1), n\} = b_i$ for each i and $A_2 = A_1$, giving the desired contradiction. Therefore, $D_M(\mathcal{I})$ contains no directed cycles. $\square$

By Propositions 2.1 and 2.2 and Lemma 2.4, $lk_{\Delta_n^2}(H)$ is homotopy equivalent to $lk_{\Delta_n^2}(H)/\mathcal{I}$. Let $\mathcal{F} = lk_{\Delta_n^2}(H) \setminus \mathcal{I}$. Then $lk_{\Delta_n^2}(H)/\mathcal{I}$ admits a cell structure with one point p_0 (corresponding to the collapsed subcomplex $\mathcal{I}$) and one k-cell for each k-simplex of $lk_{\Delta_n^2}(H)$ which lies in $\mathcal{F}$. If $G, K \in \mathcal{F}$ and G is a face of K in $lk_{\Delta_n^2}(H)$, then the boundary of K is attached to G in $lk_{\Delta_n^2}(H)/\mathcal{I}$ in the same way it is attached in $lk_{\Delta_n^2}(H)$. If $G \in \mathcal{I}$, $K \in \mathcal{F}$ then any part of the boundary of K which was attached to G in $lk_{\Delta_n^2}(H)$ is attached to p_0 in $lk_{\Delta_n^2}(H)/\mathcal{I}$.

Now $G \in \mathcal{F}$ if and only if the following conditions hold:

- $\{1, n\} \notin E(G)$ and $(G \cup H) + \{1, n\}$ is 2-connected, and
- $\{m(G), n\} \in E(H)$.

If these conditions hold then $G \cup H$ has two blocks and $m(G)$ is the unique cutpoint of $G \cup H$. It follows that if $G, K \in \mathcal{F}$ and G is a face of K in $lk_{\Delta_n^2}(H)$

then $m(K) = m(G)$ and $K \cup H$ has the same blocks as $G \cup H$. For each $i \in [n-1]$ such that $\{i, n\} \in E(H)$, let $\mathcal{L}(i, H)$ be the subcomplex of $lk_{\Delta_n^2}(H)/\mathcal{I}$ consisting of p_0 and all cells $G \in \mathcal{F}$ such that $m(G) = i$. By the argument just given, $lk_{\Delta_n^2}(H)/\mathcal{I}$ is the union of the subcomplexes $\mathcal{L}(i, H)$, and if $i \neq j$ then $\mathcal{L}(i, H) \cap \mathcal{L}(j, H) = \{p_0\}$. Thus $lk_{\Delta_n^2}(H)/\mathcal{I}$ is the wedge at p_0 of the subcomplexes $\mathcal{L}(i, H)$.

Fix i such that $\{i, n\} \in E(H)$. Define $P(i)$ to be the set of all pairs $\{A, B\}$ of subsets of $[n]$ satisfying the following conditions:

- $A \cup B = [n]$,
- $A \cap B = \{i\}$,
- $1 \in A$ and $n \in B$,
- $|A| > 1$ and $|B| > 1$, and
- if W is a block of H then either $W \subseteq A$ or $W \subseteq B$.

Note that if $G \neq p_0$ is a cell in $\mathcal{L}(i, H)$ then the two blocks A, B of $G \cup H$ satisfy the conditions just given. For $\{A, B\} \in P(i)$, let $\mathcal{L}(A, B)$ be the subcomplex of $\mathcal{L}(i, H)$ consisting of p_0 and all cells $G \in \mathcal{F}$ such that the blocks of $G \cup H$ are A, B. Then $\mathcal{L}(i, H)$ is the wedge at p_0 of the subcomplexes $\mathcal{L}(A, B)$. We now see that in order to prove Theorem 1.3(b), it suffices to show that for each i and for each $\{A, B\} \in P(i)$, the complex $\mathcal{L}(A, B)$ is homotopy equivalent to a wedge of spheres of dimension $2c(H) + b(H) - 5$. We will need the following result.

Lemma 2.5 *Let Σ be a contractible simplicial complex and let Γ be a nonempty subcomplex of Σ. Then the quotient complex Σ/Γ is homotopy equivalent to the (unreduced) suspension $susp(\Gamma)$.*

Proof: The embedding of Σ in $\Sigma \times [0, 1]$ given by $\sigma \mapsto (\sigma, 1)$ has a homotopy inverse and maps Γ to $\Gamma \times \{1\}$, so

$$\Sigma/\Gamma \simeq (\Sigma \times [0, 1])/(\Gamma \times \{1\}).$$

Since Σ is contractible, so is $\Sigma \times \{0\}$. Therefore,

$$(\Sigma \times [0, 1])/(\Gamma \times \{1\}) \simeq (\Sigma \times [0, 1])/(\Gamma \times \{1\}, \Sigma \times \{0\}).$$

Here we are using the notation for quotient spaces as given in [Bj]. Now it is straightforward to show that if we choose any point $\gamma_0 \in \Gamma$ then the map which sends (σ, x) to (γ_0, x) for $\sigma \in \Sigma$ and $x \in [0, 1]$ is a homotopy inverse for the identity embedding of $cone(\Gamma)$ into $cone(\Sigma)$, where for any space X we set $cone(X) := (X \times [0, 1])/(X \times \{0\})$. It follows that

$$(\Sigma \times [0, 1])/(\Gamma \times \{1\}, \Sigma \times \{0\}) = cone(\Sigma)/(\Gamma \times \{1\}) \simeq cone(\Gamma)/(\Gamma \times \{1\}) = susp(\Gamma).$$

□

Fix $i \in [n-1]$ with $\{i,n\} \in E(H)$ and fix $\{A,B\} \in P(i)$. Let H_A be the subgraph of H induced on A and define H_B similarly. Let $\Delta(A)$ be the simplex whose faces are (indexed by) all graphs on A and let Δ_A^2 be the subcomplex of $\Delta(A)$ consisting of all separable graphs on A. Define $\Delta(B)$ and Δ_B^2 similarly.

Since H is block-closed, each of H_A, H_B is either separable or complete. We examine the three possible cases separately.

First assume that both H_A and H_B are complete. In this case, H is a maximal face of Δ_n^2, so $lk_{\Delta_n^2}(H) = \{\emptyset\}$, that is, $lk_{\Delta_n^2}(H)$ is a (-1)-sphere. Now H is connected and has two blocks, so $2c(H) + b(H) - 5 = -1$, and Theorem 1.3(b) holds in this case.

Next assume exactly one of H_A, H_B is complete. We may assume that H_A is separable and H_B is complete. Then the cells of $\mathcal{L}(A,B)$ are p_0 along with one k-cell for each graph K on A satisfying the following conditions:

1. $|E(K)| = k+1$,

2. $K \cap H_A = \emptyset$, and

3. $K \cup H_A$ is 2-connected.

The quotient complex $lk_{\Delta(A)}(H_A)/lk_{\Delta_A^2}(H_A)$ admits a cell decomposition which is isomorphic to the given decomposition of $\mathcal{L}(A,B)$, so the two complexes are homeomorphic. Since $lk_{\Delta(A)}(H_A)$ is a simplex, Lemma 2.5 gives

$$\mathcal{L}(A,B) \simeq susp(lk_{\Delta_A^2}(H_A)).$$

Therefore, by inductive hypothesis, $\mathcal{L}(A,B)$ is homotopy equivalent to a wedge of spheres of dimension $1 + 2c(H_A) + b(H_A) - 5$. Now $c(H) = c(H_A)$ (the component of H containing B is the union of B and the component of H_A containing i), and $b(H) = b(H_A) + 1$ (the blocks of H which contain more than one vertex are those of H_A along with B). Theorem 1.3(b) now follows in this case.

Finally, assume that both H_A and H_B are separable. In this case, a cell $K \neq p_0$ in $\mathcal{L}(A,B)$ is uniquely determined by first choosing a graph K_A on A with $K_A \cap H_A = \emptyset$ and $K_A \cup H_A$ 2-connected and then choosing a graph K_B on B satisfying the same properties with respect to H_B. It follows that the given cell structure on $\mathcal{L}(A,B)$ is isomorphic with a cell structure on the quotient complex $(lk_{\Delta(A)}(H_A) * lk_{\Delta(B)}(H_B))/S$, where S is obtained from the disjoint union of $lk_{\Delta(A)}(H_A) * lk_{\Delta_B^2}(H_B)$ and $lk_{\Delta_A^2}(H_A) * lk_{\Delta(B)}(H_B)$ by identifying the subcomplexes $lk_{\Delta_A^2}(H_A) * lk_{\Delta_B^2}(H_B)$ which appear in each component of the disjoint union.

By Lemma 2.5, we have

$$\mathcal{L}(A,B) \simeq susp(S).$$

Since $lk_{\Delta(A)}(H_A)$ is a nonempty simplex, the subcomplex $T = lk_{\Delta(A)}(H_A) * lk_{\Delta^2_B}(H_B)$ of S is collapsible. By Lemma 2.2, $S \simeq S/T$. Now S/T admits a cell structure isomorphic to one admitted by

$$Q := (lk_{\Delta^2_A}(H_A) * lk_{\Delta(B)}(H_B))/(lk_{\Delta^2_A}(H_A) * lk_{\Delta^2_B}(H_B)),$$

so $S \simeq Q$. Since $lk_{\Delta(B)}(H_B)$ is a simplex which strictly contains $lk_{\Delta^2_B}(H_B)$, we may apply Lemma 2.5 to get

$$Q \simeq susp(lk_{\Delta^2_A}(H_A) * lk_{\Delta^2_B}(H_B)).$$

Now if Σ is any complex then $susp(\Sigma) \simeq S^0 * \Sigma$. Also, the join operation on complexes is associative, and $S^0 * S^0 \simeq S^1$. Combining these facts with the above observations, we get

$$\mathcal{L}(A, B) \simeq S^1 * lk_{\Delta^2_A}(H_A) * lk_{\Delta^2_A}(H_B).$$

If Σ_i is homotopy equivalent to a wedge of spheres of dimension d_i for $i \in \{1, 2\}$ then $\Sigma_1 * \Sigma_2$ is homotopy equivalent to a wedge of spheres of dimension $d_1 + d_2 + 1$. Using this fact and the inductive hypothesis, we calculate that $\mathcal{L}(A, B)$ has the homotopy type of a wedge of spheres of dimension $2(c(H_A) + c(H_B)) + b(H_A) + b(H_B) - 7$. Theorem 1.3(b) now follows, since $c(H_A) + c(H_B) = c(H) + 1$ (the component of H containing i gets counted twice in the sum) and $b(H_A) + b(H_B) = b(H)$ (each block of H which contains more than one vertex is contained in exactly one of A, B).

We will prove Corollaries 1.4 and 1.6 after recalling the following results, which appear as Theorem 5.7(i), Lemma 5.10 and Theorem 6.1 (and the following remark) in [BBLSW].

Lemma 2.6 *The lattice $\Sigma_{n,2}$ is ranked with rank function ρ, where $\rho(G) = 2n - 2c(G) - b(G)$.*

Lemma 2.7 *If the order complex of the open interval (G, K_n) in $\Sigma_{n,2}$ is topologically $(n-5)$-connected for every forest $G \in \Delta^2_n$ then $\Delta^2_{n,3}$ is homotopy equivalent to a wedge of spheres of dimension $n - 4$.*

Lemma 2.8 *Let $M_3(x) = \sum_{n \geq 3} \tilde{\chi}(\Delta^2_{n,3}) \frac{x^n}{n!}$, where $\tilde{\chi}$ is the reduced Euler characteristic. Then*

$$M_3'(x) = \ln\left(\frac{-x(x-2)}{(x-1) + \sqrt{2 - (x-1)^2}}\right).$$

To prove Corollary 1.4, we note that if G is a forest then G is block-closed with $b(G) = |E(G)|$ and $c(G) = n - |E(G)|$. It follows from Theorem 1.3 and

Proposition 1.5 that $\Delta(G, K_n)$ has the homotopy type of a wedge of spheres of dimension $2n - |E(G)| - 5$ and is therefore $(2n - |E(G)| - 6)$-connected. The first claim of Corollary 1.4 now follows from Lemma 2.7, since $|E(G)| \leq n - 1$. The second claim follows from Lemma 2.8, since the reduced Euler characteristic of a wedge of j spheres of dimension k is $(-1)^k j$.

The claim of Corollary 1.6 holds for small n by inspection, and the proof is by induction on n. Note that if H is block-closed with blocks of sizes $n_1, \ldots, n_r$ then the ideal in $\Sigma_{n,2}$ generated by H is isomorphic to the product poset $\Sigma_{n_1,2} \times \ldots \times \Sigma_{n_r,2}$. Since the product of Cohen-Macaulay posets is Cohen-Macaulay (this is well known and follows from 11.9 in [Bj]), it follows from our inductive hypothesis that if $G, H \in \Sigma_{n,2}$ with $G \subseteq H \neq K_n$ then $\Delta(G, H)$ is $(\rho(H) - \rho(G) - 3)$-connected, where ρ is the rank function of Lemma 2.6. Therefore, to prove Corollary 1.6 it suffices to show that if $G \in \Sigma_{n,2} \setminus K_n$ then $\Delta(G, K_n)$ is $(\rho(K_n) - \rho(G) - 3)$-connected. By Lemma 2.6, $\rho(K_n) - \rho(G) = 2c(G) + b(G) - 3$, and Corollary 1.6 now follows from Theorem 1.3 and Proposition 1.5.

Acknowledgments

I thank the anonymous referee, whose suggestions led to significant improvements in this paper.

References

[BBLSW] E. Babson, A. Björner, S. Linusson, J. Shareshian, V. Welker, Complexes of not i-connected graphs, *Topology*, to appear.

[Bj] A. Björner, Topological Methods, *Handbook of Combinatorics*, vol. 2, R. L. Graham, M. Grötschel and L. Lovász (editors), pp. 1819-1872, MIT Press, Cambridge, 1995.

[BW] A. Björner and J. W. Walker, A homotopy complementation formula for partially ordered sets, *European J. Combin.* **4** (1983), 11-19.

[Ch] M. K. Chari, On discrete Morse functions and combinatorial decompositions, preprint, 1996.

[Fo] R. Forman, Morse theory for cell complexes, *Advances in Math.* **134** (1998), 90-145.

[Lo] L. Lovász, *Combinatorial Problems and Exercises*, 2nd edition, North-Holland, Amsterdam, 1993.

[LW] A. Lundell and S. Weingram, *The Topology of CW Complexes*, Van Nostrand Reinhold Company, 1969.

[Qu] D. Quillen, Homotopy Properties of the Poset of Nontrivial p-Subgroups of a Group, *Advances in Math.* **28** (1978), 101-128.

[Sh] J. Shareshian, Discrete Morse theory for complexes of 2-connected graphs, preprint, 1998.

[T1] V. Turchin, Homology of complexes of biconnected graphs, *Uspekhi Mat. Nauk* **52** (1997), 189-190.

[T2] V. Turchin, Homology isomorphism of the complex of 2-connected graphs and the graph complex of trees, *Topics in Quantum Groups and Finite Type Invariants: Mathematics at the Independent University of Moscow*, B. Feigin and V. Vassiliev (editors), American Mathematical Society, 1998.

[V1] V. A. Vassiliev, Cohomology of knot spaces, *Advances in Soviet Mathematics*, vol. 1, *Theory of Singularities and its Applications*, V. I. Arnol'd (editor), pp. 23-69, American Mathematical Society, 1990.

[V2] V. A. Vassiliev, Topology of two-connected graphs and homology of spaces of knots, preprint, 1997.

[V3] V. A. Vassiliev, Invariants of Ornaments, *Singularities and Bifurcations*, V. I. Arnol'd (editor), pp.225-262, American Mathematical Society, 1994.