EULERIAN QUASISYMMETRIC FUNCTIONS

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Abstract. We introduce a family of quasisymmetric functions called *Eulerian quasisymmetric functions*, which have the property of specializing to enumerators for the joint distribution of the permutation statistics, major index and excedance number on permutations of fixed cycle type. This family is analogous to a family of quasisymmetric functions that Gessel and Reutenauer used to study the joint distribution of major index and descent number on permutations of fixed cycle type. Our central result is a formula for the generating function for the Eulerian quasisymmetric functions, which specializes to a new and surprising q-analog of a classical formula of Euler for the exponential generating function of the Eulerian polynomials. This q-analog computes the joint distribution of excedance number and major index, the only of the four important Euler-Mahonian distributions that had not yet been computed. Our study of the Eulerian quasisymmetric functions also yields results that include the descent statistic and refine results of Gessel and Reutenauer. We also obtain q-analogs, (q,p)-analogs and quasisymmetric function analogs of classical results on the symmetry and unimodality of the Eulerian polynomials. Our Eulerian quasisymmetric functions refine symmetric functions that have occurred in various representation theoretic and enumerative contexts such as in MacMahon's study of multiset derangements, in work of Procesi and Stanley on toric varieties of Coxeter complexes, in Stanley's work on symmetric chromatic polynomials, and in the work of the authors on the homology of a certain poset introduced by Björner and Welker.

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Contents

1. Introduction

Through our study [43] of the homology of a certain partially ordered set introduced by Björner and Welker, we have discovered a remarkable q-analog of a classical formula for the Eulerian polynomials. The Eulerian polynomials enumerate permutations according to their number of descents or their number of excedances. Our q-analog is a formula for the joint distribution of the excedance permutation statistic and the major index permutation statistic, the only of the four important Euler-Mahonian distributions that had not yet been addressed in the literature. Although poset topology led us to conjecture our formula, it is the theory of symmetric functions and quasisymmetric functions that provides the proof of our original formula, as well as of more refined versions involving additional permutation statistics. In the research announcement [42] these results and sketches of their proofs were presented. Here we provide the details and additional results.

The modern study of permutation statistics began with the work of Major Percy McMahon [34, Vol. I, pp. 135, 186; Vol. II, p. viii], [35]. It deals with the enumeration of permutations according to natural statistics. A permutation statistic is simply a function from the union of all the symmetric groups $\mathfrak{S}_n$ to the set of nonnegative integers. MacMahon studied four fundamental permutation statistics, the inversion index (inv), the major index (maj), the descent number (des), and the excedance number (exc), which are defined in Section 2.

MacMahon observed in [34, Vol. I, p. 186] the now well known result that the statistics des and exc are equidistributed, that is,

$$A_n(t) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}.$$

The coefficients of the polynomials $A_n(t)$ were studied by Euler, and are called *Eulerian numbers*. The polynomials are known as *Eulerian polynomials*. (Note that it is common in the literature to define the Eulerian polynomials to be $tA_n(t)$.) Any permutation statistic that is equidistributed with des and exc is called an *Eulerian statistic*. Eulerian numbers and polynomials have been extensively studied (see for example [21] and [33]). Euler proved (see [33, p. 39]) the generating function formula

$$1 + \sum_{n\geq 1} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)}-t}. \tag{1.1}$$

For a positive integer n, the polynomials $[n]_q$ and $[n]_q!$ are defined as

$$[n]_q := 1 + q + \ldots + q^{n-1}$$

and

$$[n]_q! := \prod_{j=1}^{n} [j]_q.$$

MacMahon proved in [35] the first equality in the equation

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q! = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}$$

after the second equality had been obtained in [41] by Rodrigues. A permutation statistic that is equidistributed with maj and inv is called a *Mahonian* statistic.

Much effort has been put into the examination of joint distributions of pairs of permutation statistics, one Eulerian and one Mahonian (for a sample see, [2, 3, 9, 11, 19, 22, 23, 25, 26, 27, 28, 29, 38, 39, 44, 45, 46, 57]). One beautiful result on such joint distributions is found in the paper [46] of Stanley. For permutation statistics $\mathsf{f}_1, \ldots, \mathsf{f}_k$ and a positive integer n, define the polynomial

$$A_n^{\mathsf{f}_1,\ldots,\mathsf{f}_k}(t_1, \ldots, t_k) := \sum_{\sigma\in\mathfrak{S}_n} t_1^{\mathsf{f}_1(\sigma)} t_2^{\mathsf{f}_2(\sigma)} \cdots t_k^{\mathsf{f}_k(\sigma)}.$$

Also, set

$$A_0^{\mathsf{f}_1,\ldots,\mathsf{f}_k}(t_1, \ldots, t_k) := 1.$$

Stanley showed that if we define

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}$$

then we have the q-analogue

$$\sum_{n\geq 0} A_n^{\mathrm{inv},\mathrm{des}}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}$$

of (1.1).

Although there has been much work on the joint distributions of $(\mathrm{maj},\mathrm{des})$, $(\mathrm{inv},\mathrm{exc})$ and $(\mathrm{inv},\mathrm{des})$ there are to our knowledge no results about $A_n^{\mathrm{maj},\mathrm{exc}}(q,t)$ in the existing literature prior to [42], where our work was first announced. Our main result on these polynomials is as follows. Set

$$\exp_q(z) := \sum_{n\geq 0} \frac{z^n}{[n]_q!}.$$

Theorem 1.1. *We have*

$$\sum_{n\geq 0} A_n^{\mathrm{maj},\mathrm{exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}. \tag{1.2}$$

It is well-known that $\exp_q(z)$ is a specialization (essentially the stable principal specialization) of the symmetric function

$$H(z) = H(\mathbf{x},z) := \sum_{n\geq 0} h_n(\mathbf{x})z^n,$$

where h_n is the complete homogeneous symmetric function of degree n (details are given in Section 2). Hence the right hand side of (1.2) is a specialization of the symmetric function

$$\frac{(1-tq)H(z)}{H(ztq)-tqH(z)}.$$

We introduce a family of quasisymmetric functions $Q_{n,j,k}(\mathbf{x})$, called Eulerian quasisymmetric functions, whose generating function

$$\sum_{n,j,k\geq 0} Q_{n,j,k}(\mathbf{x})t^j r^k z^n$$

specializes to

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} r^{\mathrm{fix}(\sigma)} \frac{z^n}{[n]_q!},$$

where $\mathrm{fix}(\sigma)$ is the number of fixed points of σ. The $Q_{n,j,k}$ are defined as sums of fundamental quasisymmetric functions that we associate (in a nonstandard way) with permutations $\sigma \in \mathfrak{S}_n$ satisfying $\mathrm{exc}(\sigma) = j$ and $\mathrm{fix}(\sigma) = k$. (These Eulerian quasisymmetric functions should not be confused with the quasisymmetric functions associated with Eulerian posets discussed in [4, 18].) Our central result is the following theorem.

Theorem 1.2. *We have*

$$\sum_{n,j,k\geq 0} Q_{n,j,k}(\mathbf{x}) t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)} \tag{1.3}$$

$$= \frac{H(rz)}{1-\sum_{n\geq 2} t[n-1]_t h_n z^n}. \tag{1.4}$$

The proof of Theorem 1.2 appears in Section 3. It depends on combinatorial bijections and involves nontrivial extension of techniques introduced by Gessel and Reutenauer in [28], Désarménien and Wachs in [14] and Stembridge in [51]. Gessel and Reutenauer construct a bijection from multisets of primitive circular words of fixed content to permutations in order to enumerate permutations with a given descent set and cycle type. This bijection, which is related to Stanley's theory of P-partitions [45], has also proved useful in papers of Désarménien and Wachs [13, 14], Diaconis, McGrath and Pitman [15], Hanlon [30], and Wachs [56]. Reiner [40] introduced a type B analog. Here we introduce a bicolored version of the Gessel-Reutenauer bijection.

By specializing Theorem 1.2 we get the following strengthening of Theorem 1.1.

Corollary 1.3. *We have*

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc,fix}}(q,t,r)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(rz)}{\exp_q(ztq)-tq\exp_q(z)} \tag{1.5}$$

By setting $t = 1$ in (1.5) one obtains a formula of Gessel and Reutenauer [28]. By setting $r = 0$, we obtain a new q-analog of a formula of Brenti [6, Proposition 5] for the exc-enumerator of derangements. Using Corollary 1.3, we exhibit in Section 4 explicit formulae, both recursive and closed, for $A_n^{\mathrm{maj,exc,fix}}$.

We will show that a different specialization (essentially the nonstable principal specialization) of Theorem 1.2 readily yields a further extension of Theorem 1.1 that Foata and Han obtained after seeing a preprint containing our work [42].

Corollary 1.4 (Foata and Han [20])**.** *We have*

$$\sum_{n\geq 0} A_n^{\mathrm{maj,des,exc,fix}}(q,p,t,r)\frac{z^n}{(p;q)_{n+1}} = \sum_{m\geq 0} p^m \frac{(1-qt)(z;q)_m(ztq;q)_m}{((z;q)_m - tq(ztq;q)_m)(zr;q)_{m+1}},$$

where

$$(a;q)_n := \begin{cases} 1 & \text{if } n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & \text{if } n\geq 1. \end{cases}$$

It is well-known that the Eulerian polynomials are symmetric and unimodal. In Section 5 we use Theorem 1.2 to obtain q-analogs of these results for $A_n^{\mathrm{maj,exc}}(q,q^{-1}t)$, and (q,p)-analogs for the derangement enumerator $A_n^{\mathrm{maj,des,exc,fix}}(q,p,q^{-1}t,0)$. Symmetric function analogs of these results for the Eulerian quasisymmetric functions $Q_{n,j,k}$ are also obtained.

Essential to our proof of Theorem 1.2 is a refinement of $Q_{n,j,k}$ which is quite interesting in its own right. Given a partition λ of n, we define the cycle type Eulerian quasisymmetric function $Q_{\lambda,j}$ to be the sum of the fundamental quasisymmetric functions associated with permutations that have cycle type λ and j excedances. In Section 5 we prove that $Q_{\lambda,j}$ is, in fact, a symmetric function. We also obtain another symmetry result, specifically: $Q_{\lambda,j} = Q_{\lambda,n-k-j}$ for all $\lambda \vdash n$, where k is the number of parts of λ equal to 1 and $j = 0,\dots,n-k$. By specializing this result we conclude that the cycle type Eulerian polynomials $A_{\lambda,j}^{\mathrm{maj,des,exc}}(q,p,q^{-1}t) := \sum_\sigma q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}p^{\mathrm{des}(\sigma)}t^{\mathrm{exc}(\sigma)}$, where σ ranges over all permutations that have cycle type λ and j excedances, is a symmetric polynomial in t. We conjecture that these polynomials are unimodal as well. It follows from Theorem 1.2 that the Eulerian quasisymmetric functions $Q_{n,j,k}$ are h-positive. We show that this does not hold for the more refined cycle type Eulerian quasisymmetric functions $Q_{\lambda,j}$, but conjecture that these are Schur positive. We also conjecture a symmetric function analog of unimodality for the polynomials $\sum_j Q_{\lambda,j}t^j$.

Of particular interest are the $Q_{\lambda,j}$ when the partition λ consists of a single part. In Section 6 it is observed that the $Q_{\lambda,j}$ for general λ can be expressed via plethysm in terms of these. We present a conjecture describing the coefficients of the expansion of $Q_{\lambda,j}$ in the power symmetric function basis for the case that λ consists of a single part. We also derive results on the virtual representation of the symmetric group whose Frobenius characteristic is $Q_{\lambda,j}$ for this case.

The symmetric functions $Q_{\lambda,j}$ resemble the symmetric functions L_λ studied by Gessel and Reutenauer in [28] in their work on quasisymmetric functions and permutation statistics. However our $Q_{\lambda,j}$ are not refinements of the L_λ. Indeed, L_λ is the Frobenius characteristic of a representation induced from a linear character of the centralizer of a permutation of cycle type λ. On the other hand, we show that if $\lambda = (n)$, where $n \geq 3$, then $\sum_j Q_{\lambda,j}$ is the Frobenius characteristic of a virtual representation (conjecturally, an actual representation) whose character takes nonzero values on elements that do not commute with any element of cycle type λ (see Corollary 6.8).

The symmetric function on the right hand side of (1.3) and (1.4) refines symmetric functions that have been studied earlier in the literature. Occurrences of these symmetric functions include enumerators for multiset derangements studied by MacMahon [34, Sec. III, Ch. III] and Askey and Ismail [1]; enumerators for words with no adjacent repeats studied by Carlitz, Scoville and Vaughan [10], Dollhopf, Goulden and Greene [17] and Stanley [48]; chromatic symmetric functions of Stanley [48]; and the Frobenius characteristic of the representation of the symmetric group on the degree $2j$ cohomology of the toric variety X_n associated to the Coxeter complex of the symmetric group $\mathfrak{S}_n$ studied by Procesi [37], Stanley [47], Stembridge [51, 52], and Dolgachev and Lunts [16]. It is a consequence of our work that these symmetric functions have nice interpretations as sums of fundamental quasisymmetric functions. Representations with Frobenius characteristic $Q_{n,j}$ also occur in the recent paper of the authors on poset topology [43]. Indeed, it was our study of the homology of a certain poset introduced by Björner and Welker [5] that led us to conjecture Theorems 1.1 and 1.2 in the first place. These connections and others are discussed in Section 7.

2. Permutation statistics and quasisymmetric functions

For $n \geq 1$, let $\mathfrak{S}_n$ be the symmetric group on the set $[n] := \{1, \dots, n\}$. A permutation $\sigma \in \mathfrak{S}_n$ will be represented here in two ways, either as a function that maps $i \in [n]$ to $\sigma(i)$, or in one line notation as $\sigma = \sigma_1 \dots \sigma_n$, where $\sigma_i = \sigma(i)$. If $n \leq 0$ then we set $[n] = \emptyset$ and $\mathfrak{S}_n = \{\theta\}$ where θ denotes the empty word.

The *descent set* of σ is

$$\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1}\},$$

and the *excedance set* of σ is

$$\mathrm{EXC}(\sigma) := \{i \in [n-1] : \sigma_i > i\}.$$

We now define the two basic Eulerian permutation statistics. The *descent number* and *excedance number* of σ are, respectively,

$$\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$$

and

$$\mathrm{exc}(\sigma) := |\mathrm{EXC}(\sigma)|.$$

For example, if $\sigma = 32541$, written in one line notation, then

$$\mathrm{DES}(\sigma) = \{1, 3, 4\} \qquad \text{and} \qquad \mathrm{EXC}(\sigma) = \{1, 3\};$$

hence $\mathrm{des}(\sigma) = 3$ and $\mathrm{exc}(\sigma) = 2$. If $i \in \mathrm{DES}(\sigma)$ we say that σ has a descent at i. If $i \in \mathrm{EXC}(\sigma)$ we say that $\sigma(i)$ is an excedance of σ and that i is an excedance position.

Next we define the two basic Mahonian permutation statistics. The *inversion index* of $\sigma \in \mathfrak{S}_n$ is

$$\mathrm{inv}(\sigma) := |\{(i, j) : 1 \le i < j \le n \text{ and } \sigma_i > \sigma_j\}|,$$

and the *major index* of σ is

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i,$$

For example, if $\sigma = 32541$ then $\mathrm{inv}(\sigma) = 6$ and $\mathrm{maj}(\sigma) = 8$.

We review some basic facts of Gessel's theory of quasisymmetric functions; a good reference is [50, Chapter 7]. A *quasisymmetric function* is a formal power series $f(\mathbf{x}) = f(x_1, x_2, \dots)$ of finite degree with rational coefficients in the infinitely many variables $x_1, x_2, \dots$ such that for any $a_1, \dots, a_k \in \mathbb{P}$, the coefficient of $x_{i_1}^{a_1} \dots x_{i_k}^{a_k}$ equals the coefficient of $x_{j_1}^{a_1} \dots x_{j_k}^{a_k}$ whenever $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. Thus each symmetric function is quasisymmetric.

For a positive integer n and $S \subseteq [n-1]$, define

$$F_{S,n} = F_{S,n}(\mathbf{x}) := \sum_{\substack{i_1 \ge \dots \ge i_n \ge 1 \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \dots x_{i_n}$$

and let $F_{\emptyset,0} = 1$. Each $F_{S,n}$ is a quasisymmetric function. The set $\{F_{S,n} : S \subseteq [n-1], n \in \mathbb{N}\}$ is a basis for the ring $\mathcal{Q}$ of quasisymmetric functions and $F_{S,n}$ is called *a fundamental* quasisymmetric function. If $S = \emptyset$ then $F_{S,n}$ is the complete homogeneous symmetric function h_n and if $S = [n-1]$ then $F_{S,n}$ is the elementary symmetric function e_n.

We review two important ways to specialize a quasisymmetric funtion. Let $\mathbb{Q}[q]$ denote the ring of polynomials in variable q with coefficients in $\mathbb{Q}$ and let $\mathbb{Q}[[q]]$ denote the ring of formal power series in variable q with coefficients in $\mathbb{Q}$. The *stable principal specialization* is the ring homomorphism $\Lambda : \mathcal{Q} \to \mathbb{Q}[[q]]$ defined by

$$\Lambda(x_i) = q^{i-1},$$

and the *principal specialization* of order m is the ring homomorphism $\Lambda_m : \mathcal{Q} \to \mathbb{Q}[q]$ defined by

$$\Lambda_m(x_i) = \begin{cases} q^{i-1} & \text{if } 1 \le i \le m \\ 0 & \text{if } i > m \end{cases}.$$

Sometimes we will need to apply Λ and Λ_m to quasisymmetric functions whose coefficient ring is $\mathbb{Q}$ with indeterminates adjoined.

Lemma 2.1 ([28, Lemma 5.2]). *For all $n \ge 0$ and $S \subseteq [n-1]$, we have*

$$\Lambda(F_{S,n}) = \frac{q^{\sum_{i \in S} i}}{(q;q)_n} \tag{2.1}$$

and

$$\sum_{m \ge 0} \Lambda_m(F_{S,n}) p^m = \frac{p^{|S|+1} q^{\sum_{i \in S} i}}{(p;q)_{n+1}}. \tag{2.2}$$

It follows from Lemma 2.1 that

$$\Lambda(h_n) = \frac{1}{(1-q)(1-q^2)\dots(1-q^n)} \tag{2.3}$$

for all n and therefore

$$\Lambda(H(z(1-q))) = \exp_q(z). \tag{2.4}$$

The standard way to connect quasisymmetric functions with permutation statistics is by associating the fundamental quasisymmetric funciton $F_{\mathrm{DES}(\sigma),n}$ with $\sigma \in \mathfrak{S}_n$. By Lemma 2.1, we have for any subset $A \in \mathfrak{S}_n$

$$\Lambda\left(\sum_{\sigma \in A} F_{\mathrm{DES}(\sigma),n}\right) = \frac{1}{(q;q)_n} \sum_{\sigma \in A} q^{\mathrm{maj}(\sigma)},$$

and

$$\sum_{m \ge 0} \Lambda_m \left(\sum_{\sigma \in A} F_{\mathrm{DES}(\sigma),n}\right) p^m = \frac{1}{(p;q)_{n+1}} \sum_{\sigma \in A} p^{\mathrm{des}(\sigma)+1} q^{\mathrm{maj}(\sigma)}$$

Gessel and Reutenauer [28] used this to study the (maj, des)-enumerator for permutations of a fixed cycle type.

Here we introduce a new way to associate fundamental quasisymmetric functions with permutations. For $n \in \mathbb{N}$, we set

$$[\overline{n}] := \{\overline{1}, \dots, \overline{n}\}$$

and totally order the alphabet $[n] \cup [\overline{n}]$ by

$$\overline{1} < \dots < \overline{n} < 1 < \dots < n. \tag{2.5}$$

For a permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we define $\overline{\sigma}$ to be the word over alphabet $[n] \cup [\overline{n}]$ obtained from σ by replacing σ_i with $\overline{\sigma_i}$ whenever $i \in \mathrm{EXC}(\sigma)$. For example, if $\sigma = 531462$ then $\overline{\sigma} = \overline{5}\overline{3}14\overline{6}2$. We define a descent in a word $w = w_1 \dots w_n$ over any totally ordered alphabet to be any $i \in [n-1]$ such that $w_i > w_{i+1}$ and let $\mathrm{DES}(w)$ be the set of all descents of w. Now, for $\sigma \in \mathfrak{S}_n$, we define

$$\mathrm{DEX}(\sigma) := \mathrm{DES}(\overline{\sigma}).$$

For example, $\mathrm{DEX}(531462) = \mathrm{DES}(\overline{5}\overline{3}14\overline{6}2) = \{1, 4\}$.

Lemma 2.2. *For all $\sigma \in \mathfrak{S}_n$,*

$$\sum_{i \in \mathrm{DEX}(\sigma)} i = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma), \tag{2.6}$$

and

$$|\mathrm{DEX}(\sigma)| = \begin{cases} \mathrm{des}(\sigma) & \text{if } \sigma(1) = 1 \\ \mathrm{des}(\sigma) - 1 & \text{if } \sigma(1) \neq 1 \end{cases} \tag{2.7}$$

Proof. Let

$$J(\sigma) = \{i \in [n-1] : i \notin \mathrm{EXC}(\sigma) \;\;\&\;\; i+1 \in \mathrm{EXC}(\sigma)\},$$

and let

$$K(\sigma) = \{i \in [n-1] : i \in \mathrm{EXC}(\sigma) \;\;\&\;\; i+1 \notin \mathrm{EXC}(\sigma)\}.$$

If $i \in J(\sigma)$ then $\sigma(i) \leq i < i+1 < \sigma(i+1)$. Hence $i \notin \mathrm{DES}(\sigma)$, but $i \in \mathrm{DEX}(\sigma)$. If $i \in K(\sigma)$ then $\sigma(i) \geq i+1 \geq \sigma(i+1)$. Hence $i \in \mathrm{DES}(\sigma)$, but $i \notin \mathrm{DEX}(\sigma)$. It follows that $K(\sigma) \subseteq \mathrm{DES}(\sigma)$ and

$$\mathrm{DEX}(\sigma) = \mathrm{DES}(\sigma) \uplus J(\sigma) - K(\sigma). \tag{2.8}$$

Hence

$$|\mathrm{DEX}(\sigma)| = \mathrm{des}(\sigma) + |J(\sigma)| - |K(\sigma)|. \tag{2.9}$$

Let $J(\sigma) = \{j_1 < j_2 < \dots < j_t\}$ and $K(\sigma) = \{k_1 < k_2 < \dots < k_s\}$. It is easy to see that if $\sigma(1) = 1$ then $t = s$ and

$$j_1 < k_1 < j_2 < k_2 < \dots < j_t < k_t, \tag{2.10}$$

since neither 1 nor n are excedance positions. On the other hand if $\sigma(1) \neq 1$ then $s = t + 1$ and

$$k_1 < j_1 < k_2 < j_2 < \dots < j_t < k_{t+1}. \tag{2.11}$$

It now follows from (2.9) that (2.7) holds.

To prove (2.6), we again handle the cases $\sigma(1) = 1$ and $\sigma(1) \neq 1$ separately.

Case 1: Suppose $\sigma(1) = 1$. Then (2.10) holds. By (2.8),

$$\sum_{i \in \mathrm{DEX}(\sigma)} i = \sum_{i \in \mathrm{des}(\sigma)} i - \sum_{i=1}^{t} (k_i - j_i).$$

Clearly

$$\mathrm{EXC}(\sigma) = \biguplus_{i=1}^{t} \{j_i + 1, j_i + 2, \ldots, k_i\}.$$

Hence

$$\mathrm{exc}(\sigma) = \sum_{i=1}^{t} (k_i - j_i)$$

and so (2.6) holds in this case.

Case 2: Suppose $\sigma(1) \neq 1$. Now (2.11) holds. By (2.8),

$$\sum_{i \in \mathrm{DEX}(\sigma)} i = \sum_{i \in \mathrm{des}(\sigma)} i - k_1 - \sum_{i=1}^{t} (k_{i+1} - j_i).$$

Now

$$\mathrm{EXC}(\sigma) = \{1, 2, \ldots, k_1\} \uplus \biguplus_{i=1}^{t} \{j_i + 1, j_i + 2, \ldots, k_{i+1}\},$$

which implies that

$$\mathrm{exc}(\sigma) = k_1 + \sum_{i=1}^{t} (k_{i+1} - j_i).$$

Hence (2.6) holds in this case too. □

We now define the quasisymmetric functions that play a central role in this paper. Let $n \geq 1$, $j, k \geq 0$ and $\lambda \vdash n$. The *Eulerian quasisymmetric functions* are defined by

$$Q_{n,j} = Q_{n,j}(\mathbf{x}) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma),n}(\mathbf{x}).$$

The *fixed point Eulerian quasisymmetric functions* are defined by

$$Q_{n,j,k} = Q_{n,j,k}(\mathbf{x}) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ \mathrm{fix}(\sigma) = k}} F_{\mathrm{DEX}(\sigma),n}(\mathbf{x}).$$

The *cycle type Eulerian quasisymmetric functions* are defined by

$$Q_{\lambda,j} = Q_{\lambda,j}(\mathbf{x}) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} F_{\mathrm{DEX}(\sigma),n}(\mathbf{x}),$$

where $\lambda(\sigma)$ denotes the cycle type of σ. For convenience we let $\mathrm{DEX}(\theta) = \mathrm{DES}(\theta) = \mathrm{EXC}(\theta) = \emptyset$ and $\mathrm{exc}(\theta) = \mathrm{fix}(\theta) = \mathrm{des}(\theta) = \mathrm{maj}(\theta) = 0$. So $Q_{0,0,0} = 1$. It is a consequence of Theorem 1.2 that the Eulerian quasisymmetric functions $Q_{n,j}$ and $Q_{n,j,k}$ are symmetric functions. We prove that the refinement $Q_{\lambda,j}$ is symmetric, as well, in Section 5.2.

The (q,p)-*Eulerian numbers* are defined by

$$a_{n,j}(p,q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j}} q^{\mathrm{maj}(\sigma)} p^{\mathrm{des}(\sigma)}.$$

The *fixed point* (q,p)-*Eulerian numbers* are defined by

$$a_{n,j,k}(p,q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ \mathrm{fix}(\sigma) = k}} q^{\mathrm{maj}(\sigma)} p^{\mathrm{des}(\sigma)}.$$

The *cycle type* (q,p)-*Eulerian numbers* are defined by

$$a_{\lambda,j}(p,q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} q^{\mathrm{maj}(\sigma)} p^{\mathrm{des}(\sigma)}.$$

It follows from (2.1) and(2.6) that for $\sigma \in \mathfrak{S}_n$ we have

$$\Lambda(F_{\mathrm{DEX}(\sigma),n}) = (q;q)_n^{-1} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)}.$$

Hence

$$a_{n,j,k}(q,1) = q^j (q;q)_n \Lambda(Q_{n,j,k}), \tag{2.12}$$

$$a_{\lambda,j}(q,1) = q^j (q;q)_n \Lambda(Q_{\lambda,j}). \tag{2.13}$$

From the first of these equations and (2.4), we see that Corollary 1.3 is obtained from Theorem 1.2 by applying the stable principal specialization.

The (nonstable) principal specialization is a bit more complicated. Given two partitions $\lambda \vdash n$ and $\mu \vdash m$, let (λ, μ) denote the partition of $n + m$ obtained by concatenating λ and μ and then appropriately rearranging the parts.

Lemma 2.3. *For $n, j, k \geq 0$ and $\lambda \vdash n-k$, where λ has no parts of size 1,*

$$a_{(\lambda,1^k),j}(q,p) = (p;q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im+j} \Lambda_m(Q_{(\lambda,1^{k-i}),j}).$$

Consequently, for $n, j, k \geq 0$,

$$a_{n,j,k}(q,p) = (p;q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im+j} \Lambda_m(Q_{n-i,j,k-i}).$$

Proof. For all $\sigma \in \mathfrak{S}_n$ we have, by (2.2) and Lemma 2.2,

$$\sum_{m\geq 0} \Lambda_m(F_{\mathrm{DEX}(\sigma),n}) p^m = \frac{1}{(p;q)_{n+1}} p^{|\mathrm{DEX}(\sigma)|+1} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}.$$

It follows that

$$\sum_{m\geq 0} \Lambda_m(Q_{(\lambda,1^k),j}) p^m = X^k_{\lambda,j}(q,p), \tag{2.14}$$

where

$$X^k_{\lambda,j}(q,p) := \frac{1}{(p;q)_{n+1}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j \\ \lambda(\sigma)=(\lambda,1^k)}} q^{\mathrm{maj}(\sigma)-j} p^{|\mathrm{DEX}(\sigma)|+1}.$$

By (2.7) we have

$$\begin{aligned} X^k_{\lambda,j}(q,p) := & \frac{1}{(p;q)_{n+1}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j \\ \lambda(\sigma)=(\lambda,1^k) \\ \sigma(1)=1}} q^{\mathrm{maj}(\sigma)-j} p^{\mathrm{des}(\sigma)+1} \\ & + \frac{1}{(p;q)_{n+1}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j \\ \lambda(\sigma)=(\lambda,1^k) \\ \sigma(1)\neq 1}} q^{\mathrm{maj}(\sigma)-j} p^{\mathrm{des}(\sigma)}. \end{aligned}$$

Let

$$Y^k_{\lambda,j}(q,p) := \frac{1}{(p;q)_{n+1}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j \\ \lambda(\sigma)=(\lambda,1^k) \\ \sigma(1)=1}} q^{\mathrm{maj}(\sigma)-j} p^{\mathrm{des}(\sigma)}.$$

Write $a^k_{\lambda,j}(q,p)$ for $a_{(\lambda,1^k),j}(q,p)$. We have

$$\begin{aligned}\frac{a^k_{\lambda,j}(q,p)}{q^j(p;q)_{n+1}} &= Y^k_{\lambda,j}(q,p) + X^k_{\lambda,j}(q,p) - pY^k_{\lambda,j}(q,p) \\ &= (1-p)Y^k_{\lambda,j}(q,p) + X^k_{\lambda,j}(q,p)\end{aligned} \tag{2.15}$$

Let $\varphi : \{\sigma \in \mathfrak{S}_n : \sigma(1) = 1\} \to \mathfrak{S}_{n-1}$ be the bijection defined by letting $\varphi(\sigma)$ be the permutation obtained by removing the 1 from the beginning of σ and subtracting 1 from each letter of the remaining word. It is clear that $\mathrm{maj}(\sigma) = \mathrm{maj}(\varphi(\sigma)) + \mathrm{des}(\sigma)$, $\mathrm{des}(\sigma) = \mathrm{des}(\varphi(\sigma))$, $\mathrm{exc}(\sigma) = \mathrm{exc}(\varphi(\sigma))$ and $\mathrm{fix}(\sigma) = \mathrm{fix}(\varphi(\sigma)) + 1$. Hence

$$Y^k_{\lambda,j}(q,p) = \frac{a^{k-1}_{\lambda,j}(q,qp)}{q^j(p;q)_{n+1}}$$

Plugging this into (2.15) yields the recurrence

$$\frac{a^k_{\lambda,j}(q,p)}{q^j(p;q)_{n+1}} = \frac{a^{k-1}_{\lambda,j}(q,qp)}{q^j(qp;q)_n} + X^k_{\lambda,j}(q,p).$$

By iterating this recurrence we get

$$\frac{a^k_{\lambda,j}(q,p)}{q^j(p;q)_{n+1}} = \sum_{i=0}^{k} X^{k-i}_{\lambda,j}(q,q^ip).$$

The result now follows from this and (2.14). □

Proof of Corollary 1.4. By Lemma 2.3 and Theorem 1.2, we have

$$\begin{aligned}
\sum_{n\geq 0} A_n^{\mathrm{maj,des,exc,fix}}(q,p,t,r)\frac{z^n}{(p;q)_{n+1}} \\
&= \sum_{n,j,k\geq 0} a_{n,j,k}(p,q)t^j r^k \frac{z^n}{(p;q)_{n+1}} \\
&= \sum_{n,j,k\geq 0} z^n t^j r^k \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im+j}\Lambda_m(Q_{n-i,j,k-i}) \\
&= \sum_{m\geq 0} p^m \sum_{i\geq 0}(zrq^m)^i \sum_{\substack{n,k\geq i\\ j\geq 0}} \Lambda_m(Q_{n-i,j,k-i})(qt)^j r^{k-i} z^{n-i} \\
&= \sum_{m\geq 0} \frac{p^m}{1-zrq^m}\Lambda_m \frac{(1-tq)H(zr)}{H(ztq)-tqH(z)} \\
&= \sum_{m\geq 0} p^m \frac{(1-tq)(z;q)_m(ztq;q)_m}{((z;q)_m - tq(ztq;q)_m)(zr;q)_{m+1}},
\end{aligned}$$

with the last step following from

$$\Lambda_m(H(z)) = \Lambda_m\left(\prod_{i\geq 0}\frac{1}{1-x_i z}\right) = \frac{1}{(z;q)_m}.$$

□

3. Bicolored necklaces and words

This section is devoted to the proof of Theorem 1.2. There are three main steps. In the first step (Section 3.1) we modify a bijection that Gessel and Reutenauer [28] constructed in order to enumerate permutations with a fixed descent set and fixed cycle type. This yields an alternative characterization of the Eulerian quasisymmetric functions involving bicolored necklaces. In the second step (Section 3.2) we construct a bijection from multisets of bicolored necklaces to bicolored words, which involves Lyndon decompositions of words. This yields yet another characterization of the Eulerian quasisymmetric functions. In the third step (Section 3.3) we generalize a bijection that Stembridge constructed to study the representation of the symmetric group on the cohomology of the toric variety associated with the type A Coxeter complex. This enables us to derive a recurrence relation, which yields Theorem 1.2.

3.1. **Step 1: bicolored version of the Gessel-Reutenauer bijection.** The Gessel-Reutenauer bijection is a bijection between pairs (σ, s), where σ is a permutation and s is a "compatible" weakly decreasing sequence, and multisets of primitive circular words over the alphabet of positive integers. This bijection enabled Gessel and Reutenauer to use the fundamental quasisymmetric functions $F_{\mathrm{DES}(\sigma),n}$ to study properies of permutations with a fixed descent set and cycle type. Here we introduce a bicolored version of the Gessel-Reutenauer bijection.

We consider circular words over the alphabet of bicolored positive integers

$$\mathcal{A} := \{1, \bar{1}, 2, \bar{2}, 3, \bar{3}, \dots\}.$$

(We can think of "barred" and "unbarred" as colors assigned to each positive integer.) For each such circular word and each starting position, one gets a linear word by reading the circular word in a clockwise direction. If one gets a distinct linear word for each starting position then the circular word is said to be *primitive.* For example the circular word $(\bar{1}, 1, 1)$ is primitive while the circular word $(\bar{1}, 2, \bar{1}, 2)$ is not. (If w is a linear word then (w) denotes the circular word obtained by placing the letters of w around a circle in a clockwise direction.) The *absolute value* or just *value* of a letter a is the letter obtained by ignoring the bar if there is one. We denote this by $|a|$.

Definition 3.1. A *bicolored necklace* is a primitive circular word w over alphabet $\mathcal{A}$ such that if the length of w is greater than 1 then

(1) every barred letter is followed (clockwise) by a letter less than or equal to it in absolute value
(2) every unbarred letter is followed by a letter greater than or equal to it in absolute value.

A circular word of length 1 is a bicolored necklace if its sole letter is unbarred. A *bicolored ornament* is a multiset of bicolored necklaces.

For example the following circular words are bicolored necklaces:

$$(\bar{3}, 1, 3, \bar{3}, 2, 2), (\bar{3}, 1, \bar{3}, \bar{3}, 2, 2), (\bar{3}, 1, 3, \bar{3}, \bar{2}, 2), (\bar{3}, 1, \bar{3}, \bar{3}, \bar{2}, 2), (2),$$

while $(\bar{3}, \bar{1}, 3, 3, 2, \bar{2})$ and $(\bar{3})$ are not.

From now on we will drop the word "bicolored"; so "necklace" will stand for "bicolored necklace" and "ornament" will stand for "bicolored ornament". The type $\lambda(R)$ of an ornament R is the partition whose parts are the sizes of the necklaces in R. The weight of a letter a is the indeterminate $x_{|a|}$. The weight $\mathrm{wt}(R)$ of an ornament R is the product of the weights of the letters of R. For example

$$\lambda((\bar{3}, 2, 2), (\bar{3}, \bar{2}, 1, 1, 2)) = (5, 3)$$

and

$$\mathrm{wt}((\bar{3},2,2),(\bar{3},\bar{2},1,1,2)) = x_3^2 x_2^4 x_1^2.$$

For each partition λ and nonnegative integer j, let $\mathfrak{R}_{\lambda,j}$ be the set of ornaments of type λ with j bars.

Given a permutation $\sigma \in \mathfrak{S}_n$, we say that a weakly decreasing sequence $s_1 \geq s_2 \geq \cdots \geq s_n$ of positive integers is *σ-compatible* if $s_i > s_{i+1}$ whenever $i \in \mathrm{DEX}(\sigma)$. For example, $7,7,7,5,5,4,2,2$ is σ-compatible, where $\sigma = 45162387$. Note that for all $\sigma \in \mathfrak{S}_n$,

$$F_{\mathrm{DEX}(\sigma),n} = \sum_{s_1,\dots,s_n} x_{s_1}\dots x_{s_n}$$

where $s_1,\dots,s_n$ ranges over all σ-compatible sequences.

For $\lambda \vdash n$ and $j = 0,\dots,n-1$, let $\mathrm{Com}_{\lambda,j}$ be the set of pairs (σ, s), where σ is a permutation of cycle type λ with j excedances and s is a σ-compatible sequence. Let $\phi : \mathrm{Com}_{\lambda,j} \to \mathfrak{R}_{\lambda,j}$ be the map defined by letting $\phi(\sigma, s)$ be the ornament obtained by first writing σ in cycle form with bars above the letters that are followed (cyclicly) by larger letters (i.e., the excedances) and then replacing each i with s_i, keeping the bars in place. For example, let $\sigma = 45162387$ and $s = 7,7,7,5,5,4,2,2$. First we write σ in cycle form,

$$\sigma = (1,4,6,3)(2,5)(7,8).$$

Next we put bars above the letters that are followed by larger letters,

$$(\bar{1},\bar{4},6,3)(\bar{2},5)(\bar{7},8).$$

After replacing each i by s_i, we have the ornament

$$(\bar{7},\bar{5},4,7)(\bar{7},5)(\bar{2},2).$$

Theorem 3.2. *The map $\phi : \mathrm{Com}_{\lambda,j} \to \mathfrak{R}_{\lambda,j}$ is a well-defined bijection.*

Proof. Let $(\sigma, s) \in \mathrm{Com}_{\lambda,j}$. It is clear that the circular words in the multiset $\phi(\sigma, s)$ satisfy conditions (1) and (2) of Definition 3.1 and that the number of bars of $\phi(\sigma, s)$ is j. It is also clear that $\lambda(\phi(\sigma, s)) = \lambda$.

We must now show that the circular words in $\phi(\sigma, s)$ are primitive. Suppose there is a circular word $(a_{i_1},\dots,a_{i_k})$ that is not primitive. Let this word come from the cycle $(i_1, i_2,\dots,i_k)$ of σ, where i_1 is the smallest element of the cycle. Suppose $a_{i_1},\dots,a_{i_k} = (a_{i_1},\dots a_{i_d})^{k/d}$, where $d < k$. Since $a_{i_1} = a_{i_{d+1}}$, $i_1 < i_{d+1}$, and the $|a_i|$'s form a weakly decreasing sequence, it follows that $|a_i| = |a_{i_1}|$ for all i such that $i_1 \leq i \leq i_{d+1}$. Hence since $(|a_1|, |a_2|,\dots,|a_n|)$ is σ-compatible, there can be no element of $\mathrm{DEX}(\sigma)$ in the set $\{i_1, i_1+1,\dots,i_{d+1}-1\}$. Since a_{i_1} and $a_{i_{d+1}}$ are equal, they are both barred or both unbarred. (Actually, a_{i_1} is barred since i_1 is the smallest element of its cycle, but since we repeat this argument for i_2 in the next paragraph, we don't want to use this fact.) It follows that both i_1 and i_{d+1} are excedance positions or both are not. Hence

$$\sigma(i_1) < \sigma(i_1 + 1) < \cdots < \sigma(i_{d+1}).$$

Since $\sigma(i_1) = i_2$ and $\sigma(i_{d+1}) = i_{d+2}$ we have $i_2 < i_{d+2}$. Repeated use of the above argument yields $i_3 < i_{d+3}$ and eventually $i_{k+1-d} < i_1$, which is impossible since i_1 is the smallest element of its cycle. Hence all the circular words of $\phi(\sigma, s)$ are primitive and $\phi(\sigma, s)$ is an ornament of type λ with j bars, which means that ϕ is well-defined.

To show that ϕ is a bijection, we construct its inverse $\eta : \mathfrak{R}_{\lambda,j} \rightarrow \mathrm{Com}_{\lambda,j}$, which takes the ornament R to the pair $(\sigma(R), s(R))$. Here $s(R)$ is simply the weakly decreasing rearrangment of the letters of R with bars removed. To construct the permutation $\sigma(R)$, we need to first order the alphabet $\mathcal{A}$ by

$$1 < \bar{1} < 2 < \bar{2} < \ldots. \tag{3.1}$$

Note that this ordering is different from the ordering (2.5) of the finite alphabet that was used to define DEX.

For each position x of each necklace of R, consider the infinite word w_x obtained by reading the necklace in a clockwise direction starting at position x. Let

$$w_x \leq_L w_y$$

mean that w_x is lexicographically less than or equal to w_y. We use the lexicographic order on words to order the positions: if $w_x <_L w_y$ then we say $x < y$. We break ties as follows: if $w_x = w_y$ and $x \neq y$ then x and y must be positions in distinct (but equal) necklaces since necklaces are primitive. If $w_x = w_y$ and x is a position in an earlier necklace than that of y under some fixed linear ordering of the necklaces then let $x < y$. If x is the ith *largest* position then replace the letter in position x by i. We now have a multiset of circular words in which each letter $1, 2, \ldots, n$ appears exactly once. This is the cycle form of the permutation $\sigma(R)$.

For example, if

$$R = ((\bar{7}, \bar{3}, 3, 5), (\bar{7}, 3, \bar{5}, 3), (\bar{7}, 3, \bar{5}, 3), (5)),$$

then

$$\sigma(R) = (1, 8, 13, 6), (2, 11, 4, 9), (3, 12, 5, 10), (7)$$

and

$$s(R) = 7, 7, 7, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3.$$

It is not hard to see that the letter in position x of R is barred if and only if the letter that replaces it is an excedance position of $\sigma(R)$. This implies that the number of bars of R equals the number of excedances of $\sigma(R)$.

We now show that $s(R)$ is $\sigma(R)$-compatible. Suppose

$$s(R)_i = s(R)_{i+1}.$$

We must show $i \notin \mathrm{DEX}(\sigma(R))$. Let x be the ith largest position of R and let y be the $(i+1)$st largest. Then i is placed in position x and $i+1$ is placed in position y. Let $f(w)$ denote the first letter of a word w. Then one of the following must hold

(1) $f(w_x) = s(R)_i = f(w_y)$
(2) $f(w_x) = \overline{s(R)_i} = f(w_y)$
(3) $f(w_x) = \overline{s(R)_i}$ and $f(w_y) = s(R)_i$.

Cases (1) and (2): Either $w_x >_L w_y$ or $w_x = w_y$ and x is in an earlier necklace than y. Let u be the position that follows x clockwise and let v be the position that follows y. Since w_u is the word obtained from w_x by removing its first letter, w_v is obtained from w_y by removing its first letter, and the first letters are equal, we conclude that $u > v$. Hence the letter that gets placed in position u is smaller than the letter that gets placed in position v. Since the letter placed in position u is $\sigma(R)(i)$ and the letter placed in position v is $\sigma(R)(i+1)$, we have $\sigma(R)(i) < \sigma(R)(i+1)$. Since $i, i+1 \in \mathrm{EXC}(\sigma(R))$ or $i, i+1 \notin \mathrm{EXC}(\sigma(R))$, we conclude that $i \notin \mathrm{DEX}(\sigma(R))$.

Case (3): Since the letter in position x is barred we have $i \in \mathrm{EXC}(\sigma(R))$. Since the letter in position y is not barred, we have $i+1 \notin \mathrm{EXC}(\sigma(R))$. Hence $i \notin \mathrm{DEX}(\sigma(R))$.

In all three cases we have that $s(R)_i = s(R)_{i+1}$ implies $i \notin \mathrm{DEX}(\sigma(R))$. Hence $s(R)$ is $\sigma(R)$-compatible.

Now we show that the map η is the inverse of ϕ. It is easy to see that $\phi(\eta(R)) = R$. Establishing $\eta(\phi(\sigma, s)) = (\sigma, s)$ means establishing $\sigma(\phi(\sigma, s)) = \sigma$ and $s(\phi(\sigma, s)) = s$. The latter equation is obvious. To establish the former, let $R = \phi(\sigma, s)$. Recall that R is obtained by writing σ in cycle form, barring the excedances, and then replacing each i by s_i, keeping the bars intact. Let p_i be the position that i occupied before the replacement. By ordering the cycles of σ so that the minimum elements of the cycles increase, we get an ordering of the necklaces in R, which we use to break ties between the w_p. To show that $\sigma(R) = \sigma$, we need to show that

(1) if $i < j$ then $w_{p_i} \geq_L w_{p_j}$
(2) if $i < j$ and $w_{p_i} = w_{p_j}$ then i is in a cycle whose minimum element is less than that of j.

To prove (1) and (2), we will use the following implication:

$$i < j \text{ and } w_{p_i} \leq_L w_{p_j} \implies f(w_{p_i}) = f(w_{p_j}) \text{ and } \sigma(i) < \sigma(j), \tag{3.2}$$

(Recall $f(w)$ is the first letter of a word w.)

Proof of (3.2): Since $i < j$ and s is weakly decreasing, we have $s_i \geq s_j$. Since $w_{p_i} \leq_L w_{p_j}$, we have $f(w_{p_i}) \leq f(w_{p_j})$, which implies that $s_i \leq s_j$. Hence $s_i = s_j$, which implies

$$s_i = s_{i+1} = \cdots = s_j.$$

It follows from this and the fact that s is σ-compatible that $k \notin \mathrm{DEX}(\sigma)$ for all $k = i, i+1, \ldots, j-1$. This implies that either

$$i, i+1, \ldots, j \in \mathrm{EXC}(\sigma) \text{ and } \sigma(i) < \sigma(i+1) < \cdots < \sigma(j)$$

or

$$i, i+1, \ldots, j \notin \mathrm{EXC}(\sigma) \text{ and } \sigma(i) < \sigma(i+1) < \cdots < \sigma(j)$$

or

$$i \in \mathrm{EXC}(\sigma) \text{ and } j \notin \mathrm{EXC}(\sigma).$$

In the first case, $f(w_{p_i}) = \bar{s}_i = \bar{s}_j = f(w_{p_j})$. In the second case, $f(w_{p_i}) = s_i = s_j = f(w_{p_j})$. In the third case $f(w_{p_i}) = \bar{s}_i > s_i = s_j = f(w_{p_j})$, which is impossible. Hence the conclusion of the implication (3.2) holds.

Now we use (3.2) to prove (1). Suppose $i < j$ and $w_{p_i} <_L w_{p_j}$. Then by (3.2), we have $\sigma(i) < \sigma(j)$ and $f(w_{p_i}) = f(w_{p_j})$, which implies $w_{p_{\sigma(i)}} <_L w_{p_{\sigma(j)}}$. Hence we can apply (3.2) again with $\sigma(i)$ and $\sigma(j)$ playing the roles of i and j. This yields $f(w_{p_{\sigma^2(i)}}) = f(w_{p_{\sigma^2(j)}})$, $\sigma^2(i) < \sigma^2(j)$, and $w_{p_{\sigma^2(i)}} <_L w_{p_{\sigma^2(j)}}$. Repeated application of (3.2) yields $f(w_{p_{\sigma^m(i)}}) = f(w_{p_{\sigma^m(j)}})$ for all m, which implies $w_{p_i} = w_{p_j}$, a contradiction.

Next we prove (2). Repeated application of (3.2) yields $\sigma^m(i) < \sigma^m(j)$ for all m. Hence the cycle containing i has a smaller minimum than the cycle containing j. □

Corollary 3.3. *For all $\lambda \vdash n$ and $j = 0, 1, \ldots, n-1$,*

$$Q_{\lambda,j} = \sum_{R \in \mathfrak{R}_{\lambda,j}} \mathrm{wt}(R).$$

Corollary 3.3 has several interesting consequences. For one thing, it can be used to prove that the Eulerian quasisymmetric functions $Q_{\lambda,j}$ are actually symmetric (see Section 5). It also has the following useful consequence.

Corollary 3.4. *For all n, j, k,*

$$Q_{n,j,k} = h_k Q_{n-k,j,0}.$$

It follows from Corollary 3.4 that Theorem 1.2 is equivalent to

$$\sum_{n,j\geq 0} Q_{n,j,0} t^j z^n = \frac{1}{1-\sum_{n\geq 2} t[n-1]_t h_n z^n}.$$

We rewrite this as,

$$1 = (\sum_{n,j\geq 0} Q_{n,j,0} t^j z^n)(1-\sum_{n\geq 2} t[n-1]_t h_n z^n),$$

which is equivalent to

$$\sum_{n,j\geq 0} Q_{n,j,0} t^j z^n = 1 + \sum_{n\geq 2} \sum_{\substack{j\geq 0 \\ 0\leq m\leq n-2}} Q_{m,j,0} t^{j+1}[n-m-1]_t h_{n-m}\, z^n.$$

For $n \geq 2$, we have

$$\begin{aligned}
\sum_{\substack{j\geq 0; \\ 0\leq m\leq n-2}} Q_{m,j,0} t^{j+1}[n-m-1]_t h_{n-m} &= \sum_{\substack{j\geq 0 \\ 0\leq m\leq n-2}} Q_{m,j,0} h_{n-m} \sum_{i=j+1}^{j+n-m-1} t^i \\
&= \sum_{i\geq 0} t^i \sum_{\substack{0\leq m\leq n-2 \\ i-n+m<j<i}} Q_{m,j,0} h_{n-m}
\end{aligned}$$

Hence Theorem 1.2 is equivalent to the recurrence relation

$$Q_{n,j,0} = \sum_{\substack{0\leq m\leq n-2 \\ j+m-n<i<j}} Q_{m,i,0} h_{n-m} \tag{3.3}$$

for $n \geq 2$ and $j \geq 0$.

3.2. **Step 2: banners.** In order to establish the recurrence relation (3.3), we introduce another type of configuration, closely related to ornaments.

Definition 3.5. A *banner* is a word B over alphabet $\mathcal{A}$ such that for all $i = 1, \ldots, \ell(B)$,

(1) if $B(i)$ is barred then $|B(i)| \geq |B(i+1)|$
(2) if $B(i)$ is unbarred then $|B(i)| \leq |B(i+1)|$ or $i = \ell(B)$,

where $B(i)$ denotes the ith letter of B and $\ell(B)$ denotes the length of B.

A *Lyndon word* over an ordered alphabet is a word that is strictly lexicographically larger than all its circular rearrangements. A *Lyndon factorization* of a word over an ordered alphabet is a factorization into a weakly lexicographically increasing sequence of Lyndon words. It is a result of Lyndon (see [36, Theorem 5.1.5]) that every word has a unique Lyndon factorization. The *Lyndon type* $\lambda(w)$ of a word w is the partition whose parts are the lengths of the words in its Lyndon factorization.

To apply the theory of Lyndon words to banners, we use the ordering of $\mathcal{A}$ given in (3.1). Using this order, the banner $B := \bar{2}2\bar{7}5\bar{7}\bar{5}47$ has Lyndon factorization

$$\bar{2}2 \cdot \bar{7}5 \cdot \bar{7}\bar{5}47.$$

So the Lyndon type of B is the partition $(4, 2, 2)$.

The weight $\mathrm{wt}(B)$ of a banner B is the product of the weights of its letters, where as before the weight of a letter a is $x_{|a|}$. For each partition λ and nonnegative integer j, let $\mathfrak{B}_{\lambda,j}$ be the set of banners with j bars whose Lyndon type is λ.

Theorem 3.6. *For each partition λ and nonnegative integer j, there is a weight-preserving bijection*

$$\psi : \mathfrak{B}_{\lambda,j} \to \mathfrak{R}_{\lambda,j}.$$

Consequently,

$$Q_{\lambda,j} = \sum_{B \in \mathfrak{B}_{\lambda,j}} \mathrm{wt}(B).$$

Proof. First note that there is a natural weight-preserving bijection from the set of Lyndon banners of length n to the set of necklaces of size n. To go from a Lyndon banner B to a necklace (B) simply attach the ends of B so that the left end follows the right end when read in a clockwise direction. To go from a necklace back to a banner simply find the lexicographically largest linear word obtained by reading the circular word in a clockwise direction. The number of bars of B clearly is the same as that of (B).

Let $B \in \mathfrak{B}_{\lambda,j}$ and let

$$B = B_1 \cdot B_2 \cdots B_k$$

be the unique Lyndon factorization of B. Note that each B_i is a Lyndon banner. To see this we need only check that the last letter of each B_i is unbarred. The last letter of B_k is the last letter of B; so it is clearly unbarred. For $i < k$, the last letter of B_i is strictly less than the first letter of B_i, which is less than or equal to the first letter of B_{i+1}. Since the last letter of B_i immediately precedes the first letter of B_{i+1} in the banner B, it must be unbarred.

Now define $\psi(B)$ to be the ornament whose necklaces are

$$(B_1), (B_2), \ldots, (B_k).$$

This map is clearly weight preserving, type preserving, and bar preserving. To go from an ornament back to a banner simply arrange the Lyndon banners obtained from the necklaces in the ornament in weakly increasing order and then concatenate. $\square$

3.3. **Step 3: the recurrence relation.** Define a *marked sequence* (ω, i) to be a weakly increasing finite sequence ω of positive integers together with an integer i such that $1 \leq i \leq \text{length}(\omega) - 1$. (One can visualize this as a weakly increasing sequence with a mark above any of its elements except the last.) For $n \geq 2$, let $\mathfrak{M}_n$ be the set of marked sequences of length n. For $n \geq 0$, let $\mathfrak{B}_n^0$ be the set of banners of length n whose Lyndon type has no parts of size 1. It will be convenient to consider the empty word to be a banner of length 0, weight 1, with no bars, and whose Lyndon type is the partition of 0 with no parts. So $\mathfrak{B}_0^0$ consists of a single element, namely the empty word. Note that $\mathfrak{B}_1^0$ is the empty set.

Theorem 3.6 and Theorem 3.7 below are all that is needed to establish the recurrence relation (3.3), which we have shown is equivalent to Theorem 1.2.

Theorem 3.7. *For all $n \geq 2$, there is a bijection*

$$\gamma : \mathfrak{B}_n^0 \to \biguplus_{0 \leq m \leq n-2} \mathfrak{B}_m^0 \times \mathfrak{M}_{n-m},$$

such that if $\gamma(B) = (B', (\omega, b))$ then

$$\text{wt}(B) = \text{wt}(B')\text{wt}(\omega) \tag{3.4}$$

and

$$\text{bar}(B) = \text{bar}(B') + b, \tag{3.5}$$

where $\text{bar}(B)$ *denotes the number of bars of B.*

We will make use of another type of factorization of a word over an ordered alphabet used by Désarménien and Wachs [14]. For any alphabet A, let A^+ denote the set of words over A of positive length.

Definition 3.8 ([14])**.** An *increasing* factorization of a word w of positive length over a totally ordered alphabet A is a factorization $w = w_1 \cdot w_2 \cdots w_k$ such that

(1) each w_i is of the form $a_i^{j_i} u_i$, where $a_i \in A$, $j_i > 0$ and

$$u_i \in \{x \in A : x < a_i\}^+$$

(2) $a_1 \leq a_2 \leq \cdots \leq a_k$.

For example, $87 \cdot 8866 \cdot 995587 \cdot 95$ is an increasing factorization of the word $w := 87886699558795$ over the totally ordered alphabet of positive integers. Note that this factorization is different from the Lyndon factorization of w, which is $87 \cdot 8866 \cdot 99558795$

Proposition 3.9 ([14, Lemmas 3.1 and 4.3]). *A word over an ordered alphabet admits an increasing factorization if and only if its Lyndon type has no parts of size 1. Moreover, the increasing factorization is unique.*

Proof of Theorem 3.7. Given a banner B in $\mathfrak{B}_n^0$, take its unique increasing factorization

$$B = B_1 \cdot B_2 \cdots B_k,$$

whose existence is guaranteed by Proposition 3.9. We will extract an increasing word from B_k. We have

$$B_k = a^p i_1 \cdots i_l,$$

where $p, l \geq 1$, and $a > i_1, i_2, \ldots, i_l$. It follows from the definition of banner that a is a barred letter. Determine the unique index r such that $i_1 \geq \cdots \geq i_{r-1}$ are barred, while i_r is unbarred. Then let s be the unique index that satisfies $r \leq s \leq l$ and either

(1) $i_r \leq i_{r+1} \leq \cdots \leq i_s$ are all unbarred and less than i_{r-1} (note that i_s can be equal to i_{r-1} in absolute value), while $i_{s+1} > i_{r-1}$ if $s < l$, or

(2) $i_r \leq i_{r+1} \leq \cdots \leq i_{s-1}$ are all unbarred and less than i_{r-1}, and i_s is barred and less than or equal to i_{r-1}.

Case 1: $s = l$. In this case (1) must hold since the last letter of a banner is unbarred. Let ω be the weakly increasing rearrangement of B_k with bars removed and let

$$B' = B_1 \cdot B_2 \cdots B_{k-1}. \tag{3.6}$$

To see that B' is a banner, one need only note that the last letter of B_{k-1} is unbarred (as is the last letter of each B_i). Since (3.6) is an increasing factorization of B', it follows from Proposition 3.9 that the Lyndon type of B' has no parts of size 1. Let

$$\gamma(B) = (B', (\omega, b)),$$

where b is the number of bars of B_k. Clearly $p \leq b < p + l$; so (ω, b) is a marked sequence for which (3.4) and (3.5) hold. For example, if

$$B = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}22\bar{4}2 \cdot \bar{8}\bar{8}\bar{7}\bar{5}22235.$$

then $a = \bar{8}$, $p = 2$, $r = 3$ and $s = 6 = l$. It follows that

$$(\omega, b) = (22355788, 4) \text{ and } B' = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}22\bar{4}2.$$

Case 2: $s < l$. In this case either (1) or (2) can hold. Let b be the number of bars of $i_1, i_2, \ldots, i_s$. Clearly $b < s$. If (1) holds then $r > 1$ since $i_{s+1} > i_{r-1}$; so $b > 0$. If (2) holds clearly $b > 0$. Let ω be the increasing rearrangement of $i_1, i_2, \ldots, i_s$ with bars removed, let $B'_k = a^p i_{s+1} \cdots i_l$, let

$$B' = B_1 \cdot B_2 \cdots B_{k-1} \cdot B'_k, \tag{3.7}$$

and let

$$\gamma(B) = (B', (\omega, b)).$$

Clearly B' is a banner with increasing factorization given by (3.7) and (ω, b) is a marked sequence for which (3.4) and (3.5) hold. For example, if

$$B = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}22\bar{4}2 \cdot \bar{8}\bar{8}\bar{7}\bar{5}22235\bar{6}24$$

then $a = \bar{8}$, $p = 2$, $r = 3$, and $s = 6 < l$. Hence

$$(\omega, b) = (223557, 2) \text{ and } B' = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}22\bar{4}2 \cdot \bar{8}\bar{8}\bar{6}24.$$

If

$$B = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}22\bar{4}2 \cdot \bar{8}\bar{8}\bar{7}\bar{5}223\bar{5}4\bar{6}24$$

then $a = \bar{8}$, $p = 2$, $r = 3$, and $s = 6 < l$. Hence

$$(\omega, b) = (223557, 3) \text{ and } B' = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}22\bar{4}2 \cdot \bar{8}\bar{8}4\bar{6}24.$$

In order to prove that the map γ is a bijection, we describe its inverse. Let $(B', (\omega, b)) \in \mathfrak{B}^0_m \times \mathfrak{M}_{n-m}$, where $0 \leq m \leq n-2$. Let

$$B' = B_1 \cdot B_2 \cdots B_{k-1}$$

be the unique increasing factorization of B', whose existence is guaranteed by Proposition 3.9, and let a be the largest letter of B_{k-1}. We also let ω_i denote the ith letter of ω.

Case 1: $|a| \leq \omega_{n-m}$. Let

$$B_k = \bar{\omega}_{n-m} \cdots \bar{\omega}_{n-m-b+1} \omega_1 \cdots \omega_{n-m-b}.$$

Clearly B_k is a banner with exactly b bars that are placed on a rearrangement of ω. Now let

$$B = B_1 \cdot B_2 \cdots B_{k-1} \cdot B_k.$$

It is easy to see that this is an increasing decomposition of a banner and that equations (3.4) and (3.5) hold.

Case 2: $|a| > \omega_{n-m}$. In this case we expand the banner B_{k-1} by inserting the letters of ω in the following way. Suppose

$$B_{k-1} = a^p j_1 \cdots j_l,$$

where $p, l \geq 1$, and $a > j_i$ for all i. If $j_1 > \bar{\omega}_{n-m-b+1}$ let

$$\tilde{B}_{k-1} = a^p \bar{\omega}_{n-m} \cdots \bar{\omega}_{n-m-b+1} \omega_1 \cdots \omega_{n-m-b} j_1, \dots, j_l.$$

Otherwise if $j_1 \leq \bar{\omega}_{n-m-b+1}$ let

$$\tilde{B}_{k-1} = a^p \bar{\omega}_{n-m} \cdots \bar{\omega}_{n-m-b+2} \omega_1 \cdots \omega_{n-m-b} \; \bar{\omega}_{n-m-b+1} j_1, \dots, j_l.$$

In both cases $\tilde{B}_{k-1}$ is a banner. Now let

$$B = B_1 \cdot B_2 \cdots B_{k-2} \cdot \tilde{B}_{k-1}.$$

It is easy to see that this is an increasing decomposition of a banner and that equations (3.4) and (3.5) hold. It is also easy to check that the map $(B', (\omega, b)) \mapsto B$ is the inverse of γ. $\square$

Remark 3.10. The bijection γ when restricted to banners with distinct letters (permutations) reduces to a bijection that Stembridge [51] constructed to study the representation of the symmetric group on the cohomology of the toric variety assoiciated with the type A Coxeter complex (see Section 6). For words with distinct letters the notion of decreasing decomposition coincides with the notion of Lyndon decomposition. In Stembridge's work the Lyndon decomposition corresponds to the cycle decomposition of a permutation. Although the term "marked sequence" is borrowed from Stembridge's paper, he defines the term differently from the way in which we do. However there is a close connection between his marked sequences and ours.

Remark 3.11. In Section 7 we discuss a connection, pointed out to us by Richard Stanley, between banners and words with no adjacent repeats. This connection can be used to provide an alternative to Step 3 in our proof of Theorem 1.2.

4. Alternative formulations

In this section we present some equivalent formulations of Theorem 1.2 and some immediate consequences.

Corollary 4.1 (of Theorem 1.2)**.** *Let* $Q_n(t,r) = \sum_{j,k\geq 0} Q_{n,j,k}\, t^j\, r^k$*. Then* $Q_n(t,r)$ *satisfies the following recurrence relation:*

$$Q_n(t,r) = r^n h_n + \sum_{k=0}^{n-2} Q_k(t,r) h_{n-k} t[n-k-1]_t. \tag{4.1}$$

Proof. The recurrence relation is equivalent to

$$\sum_{k=0}^{n} Q_k(t,r) h_{n-k} t[n-k-1]_t = -r^n h_n.$$

Taking the generating function we have

$$\left(\sum_{n\geq 0} Q_n(t,r) z^n\right)\left(\sum_{n\geq 0} h_n t[n-1]_t z^n\right) = -H(rz).$$

The result follows from this. ☐

The right hand side of (1.3) is the Frobenius characteristic of a graded permutation representation that Stembridge [51] described in terms of $\mathfrak{S}_n$ acting on "marked words". From his work we were led to the following formula, which can easily be proved by showing that the right hand side satisfies the recurrence relation (4.1).

Corollary 4.2. *For all* $n \geq 0$*,*

$$Q_n(t,r) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k_0 \geq 0 \\ k_1,\ldots,k_m \geq 2 \\ \sum k_i = n}} r^{k_0} h_{k_0} \prod_{i=1}^{m} h_{k_i} t[k_i - 1]_t. \tag{4.2}$$

Let

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \quad \text{and} \quad \begin{bmatrix} n \\ k_0,\ldots,k_m \end{bmatrix}_q = \frac{[n]_q!}{[k_0]_q![k_1]_q!\cdots[k_m]_q!}.$$

By taking the principal stable specialization of both sides of the recurrence relation (4.1) and the formula (4.2), we have the following result.

Corollary 4.3. *For all* $n \geq 0$*,*

$$A_n^{\mathrm{maj,exc,fix}}(q,t,r) = r^n + \sum_{k=0}^{n-2} \begin{bmatrix} n \\ k \end{bmatrix}_q A_k^{\mathrm{maj,exc,fix}}(q,t,r)\, tq[n-k-1]_{tq},$$

and

$$A_n^{\mathrm{maj,exc,fix}}(q,t,r) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k_0 \geq 0 \\ k_1,\dots,k_m \geq 2 \\ \sum k_i = n}} \begin{bmatrix} n \\ k_0,\dots,k_m \end{bmatrix}_q r^{k_0} \prod_{i=1}^{m} tq[k_i-1]_{tq}.$$

Gessel and Reutenauer [28] and Wachs [55] derive a major index q-nalaog of the classical formula for the number of derangements in $\mathfrak{S}_n$ (or more generally the number of permutations with a given number of fixed points). As an immediate consequence of [55, Corollary 3], one can obtain a $(\mathrm{maj},\mathrm{exc})$ generalization. Since this generalization also follows from Corollary 1.3 and is not explicitly stated in [55], we state and prove it here. For all $n \in \mathbb{N}$, let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$.

Corollary 4.4 (of Corollary 1.3)**.** *For all $n \geq 0$, we have*

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{fix}(\sigma)=k}} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)} = \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\sigma \in \mathcal{D}_{n-k}} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}. \tag{4.3}$$

Consequently,

$$\sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)} = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k}^{\mathrm{maj,exc}}(q,t). \tag{4.4}$$

Proof. Since the left hand side of (4.3) equals the coefficient of $r^k \frac{z^n}{[n]_q!}$ on the left hand side of (1.5), we have that the left hand side of (4.3) is $\begin{bmatrix} n \\ k \end{bmatrix}_q$ times the coefficient of $\frac{z^{n-k}}{[n-k]_q!}$ in $(1-tq)/(\exp_q(ztq)-tq\exp_q(z))$. This coefficient is precisely $\sum_{\sigma \in \mathcal{D}_{n-k}} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}$.

By summing (4.3) over all k and applying Gaussian inversion we obtain (4.4). $\square$

We point out that our results pertaining to major index have comajor index versions. We define *comajor index* of $\sigma \in \mathfrak{S}_n$ to be

$$\mathrm{comaj}(\sigma) := \sum_{i \in [n-1] \setminus \mathrm{DES}(\sigma)} i = \binom{n}{2} - \mathrm{maj}(\sigma).$$

(Note that this is different from another commonly used notion of comajor index.) For example, we have the following comajor index version of Corollary 1.3.

Corollary 4.5 (of Corollary 1.3)**.** *We have*

$$\sum_{n\geq 0} A_n^{\mathrm{comaj,exc,fix}}(q,t,r)\frac{z^n}{[n]_q!} = \frac{(1-tq^{-1})\mathrm{Exp}_q(rz)}{\mathrm{Exp}_q(ztq^{-1})-(tq^{-1})\mathrm{Exp}_q(z)} \tag{4.5}$$

Proof. Use the facts that $[n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!$ and $\exp_{q^{-1}}(z) = \mathrm{Exp}_q(z)$ to show that equations (1.5) and (4.5) are equivalent. $\square$

We also have the following comajor index version of Corollary 4.4.

Corollary 4.6. *For all $n \geq 0$, we have*

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{fix}(\sigma)=k}} q^{\mathrm{comaj}(\sigma)}t^{\mathrm{exc}(\sigma)} = q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\sigma\in\mathcal{D}_{n-k}} q^{\mathrm{comaj}(\sigma)}t^{\mathrm{exc}(\sigma)}.$$

Consequently,

$$\sum_{\sigma\in\mathcal{D}_n} q^{\mathrm{comaj}(\sigma)}t^{\mathrm{exc}(\sigma)} = \sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k}^{\mathrm{comaj,exc}}(q,t).$$

5. Symmetry and unimodality

5.1. **Eulerian quasisymmetric functions.** It is well known that the Eulerian numbers $a_{n,j}$ form a symmetric and unimodal sequence for each fixed n (see [12, p. 292]); i.e., $a_{n,j} = a_{n,n-1-j}$ for all $j = 0,1,\ldots,n-1$ and

$$a_{n,0} \leq a_{n,1} \leq \cdots \leq a_{n,\lfloor\frac{n-1}{2}\rfloor} = a_{n,\lfloor\frac{n}{2}\rfloor} \geq \cdots \geq a_{n,n-2} \geq a_{n,n-1}.$$

In this subsection we discuss symmetry and unimodality of the coefficients of t^j in the Eulerian quasisymmetric functions and the q- and (q,p)-Eulerian polynomials.

Let $f(t) := f_r t^r + f_{r+1}t^{r+1} + \cdots + f_s t^s$ be a nonzero polynomial in t whose coefficients come from a partially ordered ring R. We say that $f(t)$ is *t-symmetric* with center of symmetry $\frac{s+r}{2}$ if $f_{r+i} = f_{s-i}$ for all $i = 0,\ldots,s-r$. We say that $f(t)$ is also *t-unimodal* if

$$f_r \leq_R f_{r+1} \leq_R \cdots \leq_R f_{\lfloor\frac{s+r}{2}\rfloor} = f_{\lfloor\frac{s+r+1}{2}\rfloor} \geq_R \cdots \geq_R f_{s-1} \geq_R f_s. \tag{5.1}$$

Let Par be the set of all partitions of all nonnegative integers. By choosing a $\mathbb{Q}$-basis $b = \{b_\lambda : \lambda \in \mathrm{Par}\}$ for the space of symmetric functions, we obtain the partial order relation given by $f \leq_b g$ if $g - f$ is b-positive, where a symmetric function is said to be b-positive if it is a nonnegative linear combination of elements of the basis $\{b_\lambda\}$. Here we are concerned with the h-basis, $\{h_\lambda : \lambda \in \mathrm{Par}\}$, where $h_\lambda = h_{\lambda_1}\cdots h_{\lambda_k}$ for $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$, and the Schur basis $\{s_\lambda : \lambda \in \mathrm{Par}\}$. Since

h-positivity implies Schur-positivity, the following result also holds for the Schur basis.

Theorem 5.1. *Using the h-basis to partially order the ring of symmetric functions, we have for all n, j, k,*

(1) *the Eulerian quasisymmetric functions $Q_{n,j,k}$ and $Q_{n,j}$ are h-positive symmetric functions,*
(2) *the polynomial $\sum_{j=0}^{n-1} Q_{n,j,k}t^j$ is t-symmetric and t-unimodal with center of symmetry $\frac{n-k}{2}$,*
(3) *the polynomial $\sum_{j=0}^{n-1} Q_{n,j}t^j$ is t-symmetric and t-unimodal with center of symmetry $\frac{n-1}{2}$.*

Proof. Part (1) is a corollary of Theorem 1.2 (see also Corollary 4.2).

Parts (2) and (3) follow from Theorem 1.2 and results of Stembridge [51] on the symmetric function given on the right hand side of (1.3). For the sake of completeness, we include a proof of Parts (2) and (3) based on Stembridge's work.

It is well-known that the product of two symmetric unimodal polynomials in $\mathbb{N}[t]$ with respective centers of symmetry c_1 and c_2, is symmetric and unimodal with center of symmetry $c_1 + c_2$. This result and the proof given in [47, Proposition 1.2] hold more generally for polynomials over an arbitrary partially ordered ring. By Corollary 4.2 we have

$$\sum_{j=0}^{n-1} Q_{n,j,k}t^j = \sum_{m=0}^{\lfloor \frac{n-k}{2} \rfloor} \sum_{\substack{k_1,\dots,k_m \geq 2 \\ \sum k_i = n-k}} h_k \prod_{i=1}^{m} h_{k_i} t[k_i - 1]_t$$

Each term $h_k \prod_{i=1}^{m} h_{k_i} t[k_i - 1]_t$ is t-symmetric and t-unimodal with center of symmetry $\sum_{i \geq 0} \frac{k_i}{2} = \frac{n-k}{2}$. Hence the sum of these terms has the same property.

With a bit more effort we show that Part (3) also follows from Corollary 4.2. We have

$$\sum_{j=0}^{n-1} Q_{n,j}t^j = \sum_{j=0}^{n-1} Q_{n,j,1}t^j + \sum_{j=0}^{n-1} Q_{n,j,0}t^j + \sum_{k \geq 2} \sum_{j=0}^{n-1} Q_{n,j,k}t^j.$$

By Part (2) we need only show that

$$X(t) := \sum_{j=0}^{n-1} Q_{n,j,0}t^j + \sum_{k \geq 2} \sum_{j=0}^{n-1} Q_{n,j,k}t^j$$

is t-symmetric and t-unimodal with center of symmetry $\frac{n-1}{2}$. For any sequence of positive integers $(k_1, \dots, k_m)$, let

$$G_{k_1,\dots,k_m} := \prod_{i=1}^{m} h_{k_i} t[k_i - 1]_t.$$

We have by Corollary 4.2,

$$\sum_{j=0}^{n-1} Q_{n,j,0} t^j = \sum_{\substack{m \geq 0 \\ k_1,\dots,k_m \geq 2 \\ \sum k_i = n}} G_{k_1,\dots,k_m}$$

and

$$\sum_{k \geq 2} \sum_{j=0}^{n-1} Q_{n,j,k} t^j = \sum_{\substack{m \geq 0 \\ k_1,\dots,k_m \geq 2 \\ \sum k_i = n}} h_{k_1} G_{k_2,\dots,k_m}.$$

Hence

$$X(t) = \sum_{\substack{m \geq 0 \\ k_1,\dots,k_m \geq 2 \\ \sum k_i = n}} G_{k_1,\dots,k_m} + h_{k_1} G_{k_2,\dots,k_m}.$$

We claim that $G_{k_1,\dots,k_m} + h_{k_1} G_{k_2,\dots,k_m}$ is t-symmetric and t-unimodal with center of symmetry $\frac{n-1}{2}$. Indeed, we have

$$G_{k_1,\dots,k_m} + h_{k_1} G_{k_2,\dots,k_m} = h_{k_1}(t[k_1 - 1]_t + 1) G_{k_2,\dots,k_m}.$$

Clearly $t[k_1 - 1]_t + 1 = 1 + t + \dots t^{k_1 - 1}$ is t-symmetric and t-unimodal with center of symmetry $\frac{k_1-1}{2}$, and $G_{k_2,\dots,k_m}$ is t-symmetric and t-unimodal with center of symmetry $\frac{n-k_1}{2}$. Therefore our claim holds and $X(t)$ is t-symmetric and t-unimodal with center of symmetry $\frac{n-1}{2}$. $\square$

We now obtain analogous results for $A_n^{\mathrm{maj,des,exc,fix}}$ and $A_n^{\mathrm{maj,des,exc}}$ by applying the principal specializations. For the ring of polynomials $\mathbb{Q}[\mathbf{q}]$, where $\mathbf{q}$ is a list of indeterminates, we use the partial order relation: for $f(\mathbf{q}), g(\mathbf{q}) \in \mathbb{Q}[(\mathbf{q})]$, define $f(\mathbf{q}) \leq_{\mathbf{q}} g(\mathbf{q})$ if $g(\mathbf{q}) - f(\mathbf{q})$ has nonnegative coefficients.

Lemma 5.2. *If f is a Schur positive homogeneous symmetric function of degree n then $(q;q)_n \Lambda(f)$ is a polynomial in q with nonnegative coefficients and $(p;q)_{n+1} \sum_{m \geq 0} \Lambda_m(f) p^m$ is a polynomial in q and p with nonnegative coefficients. Consequently if f and g are homogeneous symmetric functions of degree n and $f \leq_{\mathrm{Schur}} g$ then*

$$(q;q)_n \Lambda(f) \leq_{(q)} (q;q)_n \Lambda(g)$$

and

$$(p;q)_{n+1}\sum_{m\geq 0}\Lambda_m(f)p^m \leq_{(q,p)} (p;q)_{n+1}\sum_{m\geq 0}\Lambda_m(g)p^m.$$

Proof. This follows from Lemma 2.1 and the fact that Schur functions are nonnegative linear combinations of fundamental quasisymmetric functions (cf. [50, pp. 360-361]). ☐

Theorem 5.3. *For all n,k, let*

$$A_{n,k}^{\mathrm{maj,des,exc}}(q,p,t) = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{fix}(\sigma)=k}} q^{\mathrm{maj}(\sigma)}p^{\mathrm{des}(\sigma)}t^{\mathrm{exc}(\sigma)}.$$

Then

(1) $A_{n,k}^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$ *is t-symmetric with center of symmetry* $\frac{n-k}{2}$
(2) $A_{n,0}^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$ *is t-symmetric and t-unimodal, with center of symmetry* $\frac{n}{2}$
(3) $A_{n,k}^{\mathrm{maj,des,exc}}(q,1,q^{-1}t)$ *is t-symmetric and t-unimodal, with center of symmetry* $\frac{n-k}{2}$
(4) $A_{n}^{\mathrm{maj,des,exc}}(q,1,q^{-1}t)$ *is t-symmetric and t-unimodal, with center of symmetry* $\frac{n-1}{2}$.

Proof. Since h-positivity implies Schur positivity, we can use Lemma 5.2 to specialize Theorem 5.1. By Lemma 2.3, Parts (1) and (2) are obtained by specializing Part (2) of Theorem 5.1. By (2.12), Parts (3) and (4) are obtained by specializing Parts (2) and (3) of Theorem 5.1, respectively. ☐

Remark 5.4. The $p,q=1$ case of Part (2) of Theorem 5.3 is due to Brenti [6, Corollary 1].

In Section 5.2 we conjecture that the t-symmetric polynomial $A_{n,k}^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$ is t-unimodal even when $k\neq 0$ and $p\neq 1$. However, $A_{n}^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$ is not t-symmetric as can be seen in the computation,

$$\begin{aligned} A_4^{\mathrm{maj,des,exc}}(q,p,t) = 1 &+ (3p+2pq+pq^2+2p^2q^2+2p^2q^3+p^2q^4)t\\ &+ (3p+pq+p^2q+3p^2q^2+2p^2q^3+p^3q^4)t^2\\ &+ pt^3. \end{aligned}$$

The Eulerian numbers posess a property stronger than unimodality, namely log-concavity (see [12, p. 292]). A sequence $(s_0,\dots,s_n)$ of real numbers or polynomials with real coefficients is said to be *log-concave* if $s_i^2 - s_{i-1}s_{i+1} \geq 0$ for all $i = 1, \dots, n-1$. Clearly if $(s_0, \dots, s_n)$ is log-concave and each $s_i > 0$ then $(s_0, \dots, s_n)$ is unimodal. It is natural to ask whether the unimodality results of this subsection have log-concavity versions. Using the Maple package ACE [54], we obtain

$$Q_{5,1}^2 - Q_{5,2}Q_{5,0} = h_{5,4,1} - h_{5,2,2,1} + +h_{5,3,2} + 4h_{4,3,2,1} + h_{4,4,1,1} + 4h_{3,3,2,2}.$$

Hence $Q_{5,1}^2 - Q_{5,2}Q_{5,0}$ is not h-positive. However, using ACE we have verified the following conjecture for $n \leq 8$.

Conjecture 5.5. *For all n, j, k, the symmetric functions $Q_{n,j}^2 - Q_{n,j+1}Q_{n,j-1}$ and $Q_{n,j,k}^2 - Q_{n,j+1,k}Q_{n,j-1,k}$ are Schur-positive.*

Conjecture 5.6. *For all n, k, the coefficients of the t-polynomials $A_{n,k}^{\mathrm{maj,des,exc}}(q, p, q^{-1}t)$ and $A_n^{\mathrm{maj,des,exc}}(q, 1, q^{-1}t)$ form log-concave sequences of polynomials in q and p.*

5.2. **Cycle type Eulerian quasisymmetric functions.** By means of ornaments we extend the symmetry results of Section 5.1 to the refinements $Q_{\lambda,j}$, and the *cycle type (q,p)-Eulerian polynomials* defined for each partition $\lambda \vdash n$ by

$$A_\lambda^{\mathrm{maj,des,exc}}(q, p, t) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \lambda(\sigma) = \lambda}} q^{\mathrm{maj}(\sigma)} p^{\mathrm{des}(\sigma)} t^{\mathrm{exc}(\sigma)}.$$

Theorem 5.7. *For all $\lambda \vdash n$ and $j = 0, 1, \dots, n-1$, the Eulerian quasisymmetric function $Q_{\lambda,j}$ is a symmetric function.*

Proof. For each $k \in \mathbb{P}$, we will construct an involution ψ on necklaces that exchanges the number of occurrences of the letter k (barred or unbarred) and $k+1$ (barred or unbarred) in a necklace, but preserves the number of occurrences of all other letters and the total number of bars. The result will then follow from Corollary 3.3. We start with necklaces that contain only the letters $\{k, \bar{k}, k+1, \overline{k+1}\}$. Let R be such a necklace. We may assume without loss of generality that $k = 1$. First replace all 1's with 2's and all 2's with 1's, leaving the bars in their original positions. The problem is that now each 2 that is followed by a 1 lacks a bar (call such a 2 *bad*) and each 1 that is followed by a 2 has a bar (call such a 1 *bad*). Since the number of bad 1's equals the number of bad 2's, we can move all the bars from the bad 1's to the bad 2's, thereby obtaining a necklace R' with the same number of bars as R but with the number of 1's and 2's exchanged. Let $\psi(R) = R'$. Clearly $\psi^2(R) = R$. For example if $R = (2\bar{2}1\bar{1}122\bar{2}\bar{2}11111)$ then we get $(1\bar{1}2\bar{2}211\bar{1}\bar{1}22222)$ before the bars are adjusted. After the bars are moved we have $\psi(R) = (112\bar{2}\bar{2}11\bar{1}12222\bar{2})$.

Now we handle the case in which R has (barred and unbarred) k's and $k+1$'s, and other letters which we will call intruders. The intruders enable us to form linear segments of R consisting only of (barred and unbarred) k's and $k+1$'s. To obtain such a linear segment start with a letter of value k or $k+1$ that follows an intruder and read the letters of R in a clockwise direction until another intruder is encountered. For example if

$$R = (\bar{5}334\bar{4}\bar{3}3\bar{3}36\bar{6}\bar{3}33\bar{4}244) \tag{5.2}$$

and $k = 3$ then the segments are $334\bar{4}\bar{3}3\bar{3}3$, $\bar{3}33\bar{4}$, and 44.

There are two types of segments, even segments and odd segments. An even (odd) segment contains an even (odd) number of switches, where a switch is a letter of value k followed by one of value $k+1$ or a letter of value $k+1$ followed by one of value k. We handle the even and odd segments differently. In an even segment, we replace all k's with $k+1$'s and all $k+1$'s with k's, leaving the bars in their original positions. Again, at the switches, we have bad k's and bad $k+1$'s, which are equinumerous. So we move all the bars from the bad k's to the bad $k+1$'s to obtain a good segment. This preserves the number of bars and exchanges the number of k's and $k+1$'s. For example, the even segment $334\bar{4}\bar{3}3\bar{3}3$ gets replaced by $443\bar{3}\bar{4}4\bar{4}4$ before the bars are adjusted. After the bars are moved we have $4\bar{4}33\bar{4}4\bar{4}4$.

An odd segment either starts with a k and ends with a $k+1$ or vice-verse. Both cases are handled similarly. So suppose we have an odd segment of the form

$$k^{m_1}(k+1)^{m_2}k^{m_3}(k+1)^{m_4}\dots k^{m_{2r-1}}(k+1)^{m_{2r}},$$

where each $m_i > 0$ and the bars have been suppressed. The number of switches is $2r-1$. We replace it with the odd segment

$$k^{m_2}(k+1)^{m_1}k^{m_4}(k+1)^{m_3}\dots k^{m_{2r}}(k+1)^{m_{2r-1}},$$

and put bars in their original positions. Again we may have created bad k's (but not bad $k+1$'s); so we need to move some bars around. The positions of the bad k's are in the set $\{N_1+m_2, N_2+m_4, \dots, N_r+m_{2r}\}$, where $N_i = \sum_{t=1}^{2i-2} m_t$. If there is a bar on the k in position N_i+m_{2i} we move it to position N_i+m_{2i-1}, which had been barless. This preserves the number of bars and exchanges the number of k's and $k+1$'s. For example, the odd segment $\bar{3}33\bar{4}$ gets replaced by $\bar{3}44\bar{4}$ before the bars are adjusted. After the bars are moved we have $34\bar{4}\bar{4}$.

Let $\psi(R)$ be the necklace obtained by replacing all the segments in the way described above. For example if R is the necklace given in

(5.2) then

$$\psi(R) = (\bar{5}4\bar{4}33\bar{4}4\bar{4}46\bar{6}34\bar{4}\bar{4}233).$$

It is easy to check that $\psi^2(R) = R$ for all necklaces R. Now extend the involution ψ to ornaments by applying ψ to each necklace of the ornament.

□

Theorem 5.8. *For all $\lambda \vdash n$ with exactly k parts equal to 1, and $j = 0, \ldots, n-k$,*

$$Q_{\lambda,j} = Q_{\lambda,n-k-j}.$$

Proof. We construct a type-preserving involution γ on ornaments. Let R be an ornament of type λ. To obtain $\gamma(R)$, first we bar each unbarred letter of each nonsingleton necklace of R and unbar each barred letter. Next for each i, we replace each occurrence of the ith smallest value in R with the ith largest value leaving the bars intact. Clearly $\gamma(R)$ is an ornament with $n-k-j$ bars whenever R is an ornament with j bars. Also $\gamma^2(R) = R$. The result now follows from the fact that $Q_{\lambda,j}$ is symmetric (Theorem 5.7) and from Corollary 3.3. □

Remark 5.9. Although the equation

$$Q_{n,j} = Q_{n,n-1-j} \tag{5.3}$$

follows easily from Theorem 1.2, we can give a bijective proof by means of banners, using Theorem 3.6. We construct an involution τ on $\bigcup_{\lambda \vdash n} \mathfrak{B}_{\lambda,j}$ that is similar to the involution γ used in the proof of Theorem 5.8. Let B be a banner of length n with j bars. To obtain $\tau(B)$, first we bar each unbarred letter of B, except for the last letter, and unbar each barred letter. Next for each i, we replace each occurrence of the ith smallest value in B with the ith largest value, leaving the bars intact. Clearly $\tau(B)$ is a banner of length n, with $n-1-j$ bars.

Since $Q_{n,j,k}$ and $Q_{n,j}$ are h-positive, one might expect that the same is true for the refinement $Q_{\lambda,j}$. It turns out that this is not the case as the following computation using the Maple package ACE [54] shows,

$$Q_{(6),3} = 2h_{(4,2)} - h_{(4,1,1)} + h_{(3,2,1)} + h_{(5,1)}. \tag{5.4}$$

Therefore Theorem 5.1 cannot hold for $Q_{\lambda,n}$, as stated. However, by expanding in the Schur basis we obtain

$$Q_{(6),3} = 3s_{(6)} + 3s_{(5,1)} + 3s_{(4,2)} + s_{(3,3)} + s_{(3,2,1)},$$

which establishes Schur positivity of $Q_{(6),3}$. We have used ACE [54] to confirm the following conjecture for $\lambda \vdash n$ up to $n = 8$.

Conjecture 5.10. *Let $\lambda \vdash n$. For all $j = 0, 1, \dots, n-1$, the Eulerian quasisymmetric function $Q_{\lambda,j}$ is Schur positive. Moreover, $Q_{\lambda,j} - Q_{\lambda,j-1}$ is Schur positive if $1 \le j \le \frac{n-k}{2}$, where k is the number of parts of λ that are equal to 1. Equivalently, the t-symmetric polynomial $\sum_{j=0}^{n-1} Q_{\lambda,j} t^j$ is t-unimodal under the partial order relation on the ring of symmetric functions induced by the Schur basis.*

Remark 5.11. We have also verified using ACE that $Q_{\lambda,j}^2 - Q_{\lambda,j+1}Q_{\lambda,j-1}$ is Schur-positive for all $\lambda \vdash n$ where $n \le 8$. Thus it is reasonable to conjecture that this is true for all n, j, and that log-concavity versions of the unimodality conjectures given below, hold.

From Stembridge's work [52] one obtains a nice combinatorial description of the coefficients in the Schur function expansion of $Q_{n,j,k}$. It would be interesting to do the same for the refinement $Q_{\lambda,j}$, at least when $\lambda = (n)$. In Section 6 we discuss the expansion in the power sum symmetric function basis.

By Lemmas 2.3 and 5.2, we have the following consequence of Theorem 5.8 (and refinement of Part (1) of Theorem 5.3).

Theorem 5.12. *Let λ be a partition of n with exactly k parts of size 1. Then $A_\lambda^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$ is t-symmetric with center of symmetry $\frac{n-k}{2}$.*

The $p, q = 1$ case of the following conjecture was proved by Brenti [7, Theorem 3.2].

Conjecture 5.13. *Let λ be a partition of n with exactly k parts of size 1. Then the t-symmetric polynomial $A_\lambda^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$ is t-unimodal (with center of symmetry $\frac{n-k}{2}$). Consequently for all n, k, $A_{n,k}^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$ is t-symmetric and t-unimodal with center of symmetry $\frac{n-k}{2}$.*

The next result is easy to prove and does not rely on the $Q_{\lambda,j}$.

Proposition 5.14. *Let λ be a partition of n with exactly k parts of size 1. Then for all $j = 0, 1, \dots, n-1$,*

$$a_{\lambda,j}(q,p) = a_{\lambda,n-k-j}(1/q, q^n p).$$

Proof. This follows from the type preserving involution $\phi : \mathfrak{S}_n \to \mathfrak{S}_n$ defined by letting $\phi(\sigma)$ be obtained from σ by writing σ in cycle form and replacing each i by $n-i+1$. Clearly ϕ sends permutations with j excedences to permutations with $n-k-j$ excedences. Also $i \in \mathrm{DES}(\sigma)$ if and only if $n-i \in \mathrm{DES}(\phi(\sigma))$. It follows that ϕ preserves des and

$$\mathrm{maj}(\phi(\sigma)) = \mathrm{des}(\sigma)n - \mathrm{maj}(\sigma).$$

□

By combining Proposition 5.14 and Theorem 5.12, we obtain the following result.

Corollary 5.15. *Let λ be a partition of n with exactly k parts of size 1. Then for all $j = 0, 1, \dots, n-1$,*

$$a_{\lambda,j}(q,p) = q^{2j+k-n} a_{\lambda,j}(1/q, q^n p).$$

Equivalently, the coefficient of $p^d t^j$ in $A_\lambda^{\mathrm{maj,des,exc}}(q,p,t)$ is q-symmetric with center of symmetry $j + \frac{k+nd-n}{2}$.

Conjecture 5.16. *The coefficient of $p^d t^j$ in $A_\lambda^{\mathrm{maj,des,exc}}(q,p,t)$ is q-unimodal.*

6. Further properties

The ornament characterization of $Q_{\lambda,j}$ yields a plethystic formula expressing $Q_{\lambda,j}$ in terms of $Q_{(i),k}$. (Note (i) stands for the partition with a single part i.) Let f be a symmetric function in variables $x_1, x_2, \dots$ and let g be a formal power series with positive integer coefficients. Choose any ordering of the monomials of g, where a monomial appears in the ordering m times if its coefficient is m. The *plethysm* of f and g, denoted $f[g]$ is defined to be the the formal power series obtained from f by replacing x_i by the ith monomial of g, for each i. The following result is an immediate consequence of Corollary 3.3.

Corollary 6.1. *Let λ be a partition with m_i parts of size i for all i. Then*

$$\sum_{j=0}^{|\lambda|-1} Q_{\lambda,j} t^j = \prod_{i\geq 1} h_{m_i}[\sum_{j=0}^{i-1} Q_{(i),j} t^j]. \tag{6.1}$$

Consequently,

$$\sum_{n,j\geq 0} Q_{n,j} t^j z^n = \sum_{n\geq 0} h_n[\sum_{i,j\geq 0} Q_{(i),j} t^j z^i].$$

By specializing (6.1) we obtain the following result. Recall that (λ, μ) denotes the partition of $m+n$ obtained by concatenating λ and μ and then appropriately rearranging the parts.

Corollary 6.2. *Let λ and μ be partitions of m and n, respectively. If λ and μ have no common parts then*

$$A_{(\lambda,\mu)}^{\mathrm{maj,exc}}(q,t) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q A_\lambda^{\mathrm{maj,exc}}(q,t) A_\mu^{\mathrm{maj,exc}}(q,t).$$

Consequently, if λ *has no parts equal to* 1 *then for all* j*,*

$$a_{(\lambda,1^m),j}(q,1) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q a_{\lambda,j}(q,1).$$

Corollary 6.3. *For all* j *and* k *we have*

$$Q_{(2^j,1^k),j} = h_j[h_2]h_k.$$

Moreover,

$$\sum_{\lambda,j} Q_{\lambda,j} t^j z^{|\lambda|} = \prod_{i\geq 1}(1-x_iz)^{-1} \prod_{1\leq i\leq j}(1-x_ix_jtz^2)^{-1}, \tag{6.2}$$

where λ *ranges over all partitions with no parts greater than* 2*.*

By specializing (6.2) one can obtain formulas for the generating functions of the $(\mathrm{maj}, \mathrm{des}, \mathrm{exc})$ and the $(\mathrm{maj}, \mathrm{exc})$ enumerators of involutions. Since exc and fix determine each other for involutions, these formulas can be immediately obtained from the formulas that Gessel and Reutenauer derived for $\mathrm{maj}, \mathrm{des}, \mathrm{fix}$ in [28, Theorem 7.1].

It follows from Corollary 6.1 that in order to prove the conjecture on Schur positivity of $Q_{\lambda,j}$ (Conjecture 5.10), it suffices to prove it for all λ with a single part because the plethysm of Schur positive symmetric functions is a Schur positive symmetric function. Note that if λ has no parts greater than 2, then by Corollary 6.3, we conclude that $Q_{\lambda,j}$ is Schur positive.

The Frobenius characteristic is a fundamental isomorphism from the ring of virtual representations of the symmetric groups to the ring of symmetric functions over the integers. We recall the definition here. For a virtual representation V of $\mathfrak{S}_n$, let χ^V_λ denote the value of the character of V on the conjugacy class of type λ. If λ has m_i parts of size i for each i define

$$z_\lambda := 1^{m_1}m_1!2^{m_2}m_2!\cdots n^{m_n}m_n!.$$

Let

$$p_\lambda := p_{\lambda_1}\cdots p_{\lambda_k},$$

for $\lambda = (\lambda_1, \dots, \lambda_k)$, where p_n is the power symmetric function $\sum_{i\geq 0} x_i^n$. The Frobenius characteristic of a virtual representation V of $\mathfrak{S}_n$ is defined as follows:

$$\mathrm{ch}V := \frac{1}{n!}\sum_{\lambda\vdash n} z_\lambda\ \chi^V_\lambda\ p_\lambda.$$

Recall that the Frobenius characteristic of a virtual representation of $\mathfrak{S}_n$ is h-positive if and only if it is a permutation representation induced from Young subgroups. Hence by part (1) of Theorem 5.1, the Eulerian quasisymmetric function $Q_{n,j,k}$ is the Frobenius characteristic of a permutation representation induced from Young subgroups. However (5.4) shows that this is not the case in general for the refined Eulerian quasisymmetric functions $Q_{\lambda,j}$. (It can be shown that $Q_{(6),3}$ is not the Frobenius characteristic of any permutation representation at all.) Recall also that the Frobenius characteristic of a virtual representation is Schur positive if and only if it is an actual representation. Hence if $V_{\lambda,j}$ is the virtual representation whose Frobenius characteristic is $Q_{\lambda,j}$ then Conjecture 5.10 says that $V_{\lambda,j}$ is an actual representation.

Proposition 6.4. *Let $\lambda \vdash n$. Then the dimension of $V_{\lambda,j}$ equals the number of permutations of cycle type λ with j excedances. Moreover, the dimension of $V_{(n),j}$ is the Eulerian number $a_{n-1,j-1}$.*

Proof. The dimension of $V_{\lambda,j}$ is the coefficient of $x_1x_2\cdots x_n$ in $Q_{\lambda,j}$. Using the definition of $Q_{\lambda,j}$, this is the number of permutations of cycle type λ with j excedances.

The set of n-cycles with j excedances maps bijectively to the set $\{\sigma \in \mathfrak{S}_{n-1} : \mathrm{des}(\sigma) = j-1\}$. Indeed, the bijection is obtained by writing the cycle in the form $(c_1, \dots, c_{n-1}, n)$ and then extracting the word $c_{n-1}\cdots c_1$. Hence the number of n-cycles with j excedances is the Eulerian number $a_{n-1,j-1}$. □

We have a conjecture for the character of $V_{(n),j}$, which generalizes Proposition 6.4. We have confirmed our conjecture up to $n = 8$ using the Maple package ACE [54]. Equivalently, our conjecture gives the coefficients in the expansion of $Q_{(n),j}$ in the basis of power symmetric functions and implies that $Q_{(n),j}$ is p-positive. For a polynomial $F(t) = a_0 + a_1t + ... + a_kt^k$ and a positive integer m, define $F(t)_m$ to be the polynomial obtained from $F(t)$ by erasing all terms a_it^i such that $\gcd(m, i) \neq 1$. For example, if $F(t) = 1 + t + 2t^2 + 3t^3$ then $F(t)_2 = t + 3t^3$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, define

$$g(\lambda) := \gcd(\lambda_1, ..., \lambda_k)$$

.

Conjecture 6.5. *For $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, let*

$$G_\lambda(t) := \left(tA_{k-1}(t) \prod_{i=1}^{k} [\lambda_i]_t \right)_{g(\lambda)}.$$

Then

$$\sum_{j=0}^{n-1} Q_{(n),j} t^j = \frac{1}{n!} \sum_{\lambda \vdash n} z_\lambda G_\lambda(t) p_\lambda.$$

Equivalently, the character of $V_{(n),j}$ evaluated on conjugacy class λ is the coefficient of t^j in $G_\lambda(t)$.

Conjecture 6.5 holds whenever $\lambda_k = 1$, see Corollary 6.8. It follows from this that the character of $V_{(n),j}$ evaluated at $\lambda = 1^n$ is the Eulerian number $a_{n-1,j-1}$. This special case of the conjecture is also a consequence of Proposition 6.4 because any character evaluated at 1^n is the dimension of the representation. In the tables below we give the character values for $V_{(n),j}$ at conjugacy class λ, which were computed using ACE [54] and confirm the conjecture up to $n = 8$. The columns are indexed by n, j and the rows by the partitions λ.

	$4,1$	$4,2$
4	1	0
31	1	1
2^2	1	0
21^2	1	2
1^4	1	4

	$5,1$	$5,2$
5	1	1
41	1	1
32	1	2
31^2	1	2
2^21	1	3
21^3	1	5
1^5	1	11

	$6,1$	$6,2$	$6,3$
6	1	0	0
51	1	1	1
42	1	0	2
3^2	1	2	0
41^2	1	2	2
321	1	3	4
2^3	1	0	6
31^3	1	5	6
2^21^2	1	6	10
21^4	1	12	22
1^6	1	26	66

	7,1	7,2	7,3
7	1	1	1
61	1	1	1
52	1	2	2
43	1	2	3
51^2	1	2	2
421	1	3	4
3^21	1	3	5
32^2	1	4	7
41^3	1	5	6
321^2	1	6	11
2^31	1	7	16
31^4	1	12	23
2^21^3	1	13	34
21^5	1	27	92
1^7	1	57	302

	8,1	8,2	8,3	8,4
8	1	0	1	0
71	1	1	1	1
62	1	0	2	0
53	1	2	3	3
4^2	1	0	3	0
61^2	1	2	2	2
521	1	3	4	4
431	1	3	5	6
42^2	1	0	7	0
3^22	1	4	8	10
51^3	1	5	6	6
421^2	1	6	11	12
3^21^2	1	6	12	16
32^21	1	7	17	22
2^4	1	0	23	0
41^4	1	12	23	24
321^3	1	13	35	46
2^31^2	1	14	47	68
31^5	1	27	93	118
2^21^4	1	28	119	184
21^6	1	58	359	604
1^8	1	120	1191	2416

Conjecture 6.5 resembles the following immediate consequence of Theorem 1.2 and Stembridge's computation [51, Proposition 3.3] of the coefficients in the expansion of the $r = 1$ case of the right hand side of (1.3), in the basis of power symmetric functions.

Proposition 6.6. *We have*

$$\sum_{j=0}^{n-1} Q_{n,j}t^j = \frac{1}{n!}\sum_{\lambda\vdash n} z_\lambda \left(A_{\ell(\lambda)}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right) p_\lambda,$$

where $\ell(\lambda)$ denotes the length of the partition λ and λ_i denotes its ith part.

By using banners we are able to prove the following curious fact about the representation $V_{(n),j}$.

Theorem 6.7. *For all $j = 0, \ldots, n-1$, the restriction of $V_{(n),j}$ to $\mathfrak{S}_{n-1}$ is the permutation representation whose Frobenius characteristic is $Q_{n-1,j-1}$.*

Proof. Given a homogeneous symmetric function f of degree n, let $\tilde{f}$ be the polynomial obtained from f by setting $x_i = 0$ for all $i > n$. Since f is symmetric, $\tilde{f}$ determines f.

We use the fact that for any virtual representation V of $\mathfrak{S}_n$,

$$\widetilde{\operatorname{ch} V\!\downarrow_{\mathfrak{S}_{n-1}}} = \frac{\partial}{\partial x_n} \widetilde{\operatorname{ch} V}\,|_{x_n=0}. \tag{6.3}$$

By Theorem 3.6, $\widetilde{Q_{(n),j}}$ is the sum of weights of Lyndon banners of length n, with j bars, whose letters have value at most n. The partial derivative with respect to x_n of the weight of a such a Lyndon banner B is 0 unless n or $\bar{n}$ appears in B. Since B is Lyndon, $\bar{n}$ must be its first letter. If the partial derivative is not 0 after setting $x_n = 0$ then all the other letters of B must be less in absolute value than n. In this case, the partial derivative is the weight of the banner B' obtained from B by removing its first letter $\bar{n}$. We thus have

$$\frac{\partial}{\partial x_n} \widetilde{\operatorname{ch} V_{(n),j}}|_{x_n=0} = \sum_{B'} \operatorname{wt}(B'), \tag{6.4}$$

where B' ranges over the set of all banners obtained by removing the first letter from a Lyndon banner of length n with j bars, whose first letter is $\bar{n}$ and whose other letters have value strictly less than n. Clearly this is the set of all banners of length $n-1$, with $j-1$ bars, whose letters have value at most $n-1$. Thus the sum on the right hand side of (6.4) is precisely $\widetilde{Q_{n-1,j-1}}$. It therefore follows from (6.3) that

$$\widetilde{\operatorname{ch} V_{(n),j}\!\downarrow_{\mathfrak{S}_{n-1}}} = \widetilde{Q_{n-1,j-1}},$$

which implies

$$\operatorname{ch} V_{(n),j}\!\downarrow_{\mathfrak{S}_{n-1}} = Q_{n-1,j-1},$$

□

Theorem 6.7, along with Proposition 6.6, allows us to prove that Conjecture 6.5 holds when λ has a part of size one.

Corollary 6.8. *Let $\lambda = (\lambda_1, \dots, \lambda_k = 1)$ be a partition of n and let $\sigma \in \mathfrak{S}_n$ have cycle type λ. Then the character of $V_{(n),j}$ evaluated at σ is the coefficient of t^j in $tA_{k-1}(t) \prod_{i=1}^{k} [\lambda_i]_t$.*

Proof. We may assume that σ fixes n, which alllows us to think of σ as an element of $\mathfrak{S}_{n-1}$. Let $V_{n-1,j-1}$ be the representation of $\mathfrak{S}_{n-1}$ whose Frobenius characteristic is $Q_{n-1,j-1}$. By Theorem 6.7, the character value in question is equal to the character value of σ on $V_{n-1,j-1}$. Corollary 6.8 now follows from Proposition 6.6. □

7. Other occurences

The Eulerian quasisymmetric functions refine symmetric functions that have appeared earlier in the literature. A multiset derangement of order n is a $2 \times n$ array of positive integers whose top row is weakly increasing, whose bottom row is a rearrangement of the top row, and whose columns contain distinct entries. An excedance of a multiset derangement $D = (d_{i,j})$ is a column j such that $d_{1,j} < d_{2,j}$. Given a multiset derangement $D = (d_{i,j})$, let $x^D := \prod_{i=1}^n x_{d_{1,i}}$. For all $j < n$, let $\mathcal{D}_{n,j}$ be the set of all derangements in $\mathfrak{S}_n$ with j excedances and let $\mathcal{MD}_{n,j}$ be the set of all multiset derangements of order n with j excedances. Set

$$d_{n,j}(\mathbf{x}) := \sum_{D \in \mathcal{MD}_{n,j}} \mathrm{x}^D.$$

Askey and Ismail [1] (see also [32]) proved the following t-analog of MacMahon's [34, Sec. III, Ch. III] result on multiset derangements

$$\sum_{j,n \geq 0} d_{n,j}(\mathbf{x}) t^j z^n = \frac{1}{1 - \sum_{i \geq 2} t[i-1]_t e_i z^i}. \tag{7.1}$$

Corollary 7.1 (to Theorem 1.2)**.** *For all $n, j \geq 0$ we have*

$$d_{n,j}(\mathbf{x}) = \omega Q_{n,j,0} = \sum_{\sigma \in \mathcal{D}_{n,j}} F_{[n-1] \setminus \mathrm{DEX}(\sigma), n}, \tag{7.2}$$

where ω is the standard involution on the ring of symmetric functions, which takes h_n to e_n. Consequenty,

$$d_{n,j}(1, q, q^2, \dots) = (q;q)_n^{-1} \sum_{\sigma \in \mathcal{D}_{n,j}} q^{\mathrm{comaj}(\sigma)+j}, \tag{7.3}$$

and

$$\sum_{m \geq 0} p^m \Lambda_m d_{n,j}(\mathbf{x}) = (p;q)_{n+1}^{-1} \sum_{\sigma \in \mathcal{D}_{n,j}} q^{\mathrm{comaj}(\sigma)+j} p^{n-\mathrm{des}(\sigma)+1}. \tag{7.4}$$

Proof. The right hand side of (7.1) can be obtained by applying ω to the right hand side of (1.4) and setting $r = 0$. Hence the first equation of (7.2) holds.

The involution on the ring $\mathcal{Q}$ of quasisymmetric funcitons defined by $F_{S,n} \mapsto F_{[n-1] \setminus S, n}$ restricts to ω on the ring of symmetric functions (cf. [50, Exercise 7.94.a]). Hence the second equation of (7.2) follows from the definition of $Q_{n,j,0}$. Equations (7.3) and (7.4) now follow from Lemmas 2.1 and 2.2. $\square$

Let W_n be the set of all words of length n over alphabet $\mathbb{P}$ with no adjacent repeats, i.e.,

$$W_n := \{w \in \mathbb{P}^n : w(i) \neq w(i+1)\ \forall i = 1, 2, \dots, n-1\}.$$

Define the enumerator

$$Y_n(x_1, x_2, \dots) := \sum_{w \in W_n} \mathbf{x}^w,$$

where $\mathbf{x}^w := x_{w(1)} \cdots x_{w(n)}$. In [10] Carlitz, Scoville and Vaughan prove the identity

$$\sum_{n \geq 0} Y_n(\mathbf{x}) z^n = \frac{\sum_{i \geq 0} e_i z^i}{1 - \sum_{i \geq 2} (i-1) e_i z^i}. \tag{7.5}$$

(See Dollhopf, Goulden and Greene [17] and Stanley [48] for alternative proofs.) It was observed by Stanley that there is a straightforward generalization of (7.5).

Theorem 7.2 (Stanley (personal communication))**.** *For all $j < n$, define*

$$Y_{n,j}(x_1, x_2, \dots) := \sum_{\substack{w \in W_n \\ \operatorname{des}(w) = j}} \mathbf{x}^w.$$

Then

$$\sum_{n,j \geq 0} Y_{n,j}(\mathbf{x}) t^j z^n = \frac{(1-t)E(z)}{E(zt) - tE(z)}, \tag{7.6}$$

where

$$E(z) = \sum_{n \geq 0} e_n z^n.$$

By combining Theorems 1.2 and 7.2 we conclude that

$$Q_{n,j} = \omega Y_{n,j}.$$

Stanley (personal communication) observed that there is a combinatorial interpretation of this identity in terms of P-partition reciprocity [49, Section 4.5]. Indeed, words in W_n with fixed descent set $S \subseteq [n-1]$ can be identified with strict P-partitions where P is the poset on $\{p_1, \dots, p_n\}$ generated by cover relations $p_i < p_{i+1}$ if $i \in S$ and $p_{i+1} < p_i$ if $i \notin S$. Banners of length n in which the set of positions of barred letters equals S can be identified with P-partitions for the same poset P. It therefore follows from P-partition reciprocity that

$$Y_{n,j}(\mathbf{x}) = \omega \sum_{B} \operatorname{wt}(B), \tag{7.7}$$

summed over all banners of length n with j bars.

In [48] Stanley views words in W_n as proper colorings of a path P_n with n vertices and Y_n as the chromatic symmetric function of P_n. The chromatic symmetric function of a graph $G = (V, E)$ is a symmetric function analog of the chromatic polynomial χ_G of G. Stanley [48, Theorem 4.2] also defines a symmetric function analog of $(-1^{|V|})\chi_G(-m)$ which enumerates all pairs (η, c) where η is an acyclic orientation of G and $c : V \to [m]$ is a coloring satisfying $c(u) \leq c(v)$ if (u, v) is an edge of η. For $G = P_n$, one can see that these pairs can be identified with banners of length n. Hence Stanley's reciprocity theorem for chromatic symmetric functions [48, Theorem 4.2] reduces to an identity that is refined by (7.7) when $G = P_n$.

Another interesting combinatorial interpretation of the Eulerian quasisymmetric functions comes from Gessel (personal communication). He considers the set U_n of words of length n over the alphabet $\mathbb{P}$ with no double (i.e., adjacent) descents and no descent in the last position $n - 1$; and proves

$$\sum_{n\geq 0} z^n \sum_{w\in U_n} \mathbf{x}^w t^{\mathrm{des}(w)}(1+t)^{n-1-2\mathrm{des}(w)} = \frac{(1-t)H(z)}{H(zt) - tH(z)}, \tag{7.8}$$

where as above $\mathbf{x}^w := x_{w(1)} \cdots x_{w(n)}$.

The symmetric function on the right hand side of (7.8) has also occurred in the work of Processi [37], Stanley [47], Stembridge [51, 52], and Dolgachev and Lunts [16]. They studied a representation of the symmetric group on the cohomology of the toric variety X_n associated with the Coxeter complex of $\mathfrak{S}_n$. (See, for example [8], for a discussion of Coxeter complexes and [24] for an explanation of how toric varieties are associated to polytopes.) The action of $\mathfrak{S}_n$ on the Coxeter complex determines an action on X_n and thus a linear representation on the cohomology groups of X_n. As X_n is a complex manifold (of dimension $n - 1$), $H^d(X_n) = 0$ whenever d is odd. The action of $\mathfrak{S}_n$ on X_n induces a representation of $\mathfrak{S}_n$ on the cohomology $H^{2j}(X_n)$ for each $j = 0, \ldots, n-1$. Stanley [47], using a formula of Procesi [37], proves that

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} \mathrm{ch} H^{2j}(X_n)\, t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)},$$

where ch denotes the Frobenius characteristic Combining this with Theorem 1.2 yields the following conclusion.

Theorem 7.3. *For all $j = 0, 1, \ldots, n-1$,*

$$\mathrm{ch} H^{2j}(X_n) = Q_{n,j}.$$

Finally, we discuss the occurrence of the Eulerian quasisymmetric functions and the q-Eulerian polynomials that motivated the work in this paper. This involves the homology of certain partially ordered sets (posets). Recall that for a poset P, the *order complex* ΔP is the abstract simplicial complex whose vertices are the elements of P and whose k-simplices are totally ordered subsets of size $k+1$ from P. The (reduced) homology of P is given by $\tilde{H}_k(P) := \tilde{H}_k(\Delta P; \mathbb{C})$. (See [58] for a discussion of poset homology.)

The first poset we consider is the Rees product $(B_n \setminus \{\emptyset\}) * C_n$, where B_n is the Boolean algebra on $\{1, 2, \dots, n\}$ and C_n is an n-element chain. Rees products of posets were introduced by Björner and Welker in [5], where they study connections between poset topology and commutative algebra. (Rees products of affine semigroup posets arise from the ring-theoretic Rees construction.) The Rees product of two ranked posets is a subposet of the usual product poset (see [5] or [43] for the precise definition.).

Björner and Welker [5] conjectured and Jonsson [31] proved that the dimension of the top homology of $(B_n \setminus \{\emptyset\}) * C_n$ is equal to the number of derangements in $\mathfrak{S}_n$. Below we discuss equivariant versions and q-analogs of both this result and refinements, which were derived in [43] using results of this paper.

The poset $(B_n \setminus \{\emptyset\}) * C_n$ has n maximal elements all of rank n. The usual action of $\mathfrak{S}_n$ on B_n induces actions of $\mathfrak{S}_n$ on $(B_n \setminus \{\emptyset\}) * C_n$ and on each lower order ideal generated by a maximal element, which respectively induce representations of $\mathfrak{S}_n$ on the homology of $(B_n \setminus \{\emptyset\}) * C_n$ and on the homology of each open lower order ideal.

Theorem 7.4 ([43]). *Let $x_1, \dots, x_n$ be the maximal elements of $(B_n \setminus \{\emptyset\}) * C_n$. For each $j = 1, \dots, n$, let $I_{n,j}$ be the open lower order ideal generated by x_j. Then $\dim \tilde{H}_{n-2}(I_{n,j})$ equals the Eulerian number $a_{n,j-1}$ and*

$$\mathrm{ch}(\tilde{H}_{n-2}(I_{n,j})) = \omega Q_{n,j-1}.$$

In [43] we derive the following equivariant version of the Björner-Welker-Jonsson result as a cosequence of Theorem 7.4,

$$\mathrm{ch}(\tilde{H}_{n-1}((B_n \setminus \{\emptyset\}) * C_n)) = \omega \sum_{j=0}^{n-1} Q_{n,j,0}.$$

A q-analog of Theorem 7.4 is obtained by considering the Rees product $(B_n(q) \setminus \{0\}) * C_n$, where $B_n(q)$ is the lattice of subspaces of the n-dimensional vector space $\mathbb{F}_q^n$ over the q element field $\mathbb{F}_q$. Like

$(B_n \setminus \{\emptyset\}) * C_n$, the q-analog $(B_n(q) \setminus \{0\}) * C_n$ has n maximal elements all of rank n.

Theorem 7.5 ([43]). *Let $x_1, \dots, x_n$ be the maximal elements of $(B_n(q) \setminus \{0\}) * C_n$. For each $j = 1, \dots, n$, let $I_{n,j}(q)$ be the open lower order ideal generated by x_j. Then*

$$\dim \tilde{H}_{n-2}(I_{n,j}(q)) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j-1}} q^{\mathrm{comaj}(\sigma)+j-1},$$

where $\mathrm{comaj}(\sigma) = \binom{n}{2} - \mathrm{maj}(\sigma)$.

As a consequence, in [43] we obtain the following q-analog of the Björner-Welker-Jonsson result:

$$\dim \tilde{H}_{n-1}((B_n(q) \setminus \{0\}) * C_n) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{comaj}(\sigma)+\mathrm{exc}(\sigma)},$$

where $\mathcal{D}_n$ is the set of derangements in $\mathfrak{S}_n$. In [43] we also derive type BC analogs (in the sense of Coxeter groups) of the Björner-Welker-Jonsson derangement result and our q-analog.

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