Discrete Morse theory for complexes of 2-connected graphs

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Abstract

Using the discrete Morse theory of R. Forman, we find a basis for the unique nonzero homology group of the complex of 2-connected graphs on n vertices. This answers a question of V. Vassiliev which arises in his study of knot invariants.

1 Introduction

Let X be a set of size $n > 1$, which will be identified from now on with the set $[n] = \{1, \dots, n\}$. The collection of all subsets of the set $\binom{X}{2}$ of 2-sets from X forms an abstract simplicial complex $\Delta(n)$, whose geometric realization is the simplex of dimension $\binom{n}{2} - 1$. An m-dimensional face F of $\Delta(n)$ is identified with the graph $G = G(F)$ on vertex set X whose edges are the $m + 1$ elements of F. From now on, for any subcomplex Σ of $\Delta(n)$ and any face F of Σ, the face F and the graph $G = G(F)$ will both be denoted by G.

A graph $G = (V, E)$ on vertex set $V = V(G)$ with edge set $E = E(G) \subseteq \binom{V}{2}$ is called **connected** if for each pair v, w of distinct vertices in V there exists a **path** $\{v = v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{r-1}, v_r = w\}$ in $E(G)$. Such a path will often be denoted by $v_0, v_1, \dots, v_r$. We will write $H \subseteq G$ when G, H are graphs with $V(H) = V(G)$ and $E(H) \subseteq E(G)$. For $S \subseteq V$, the subgraph of G **induced** on S is the graph (S, E_S), where E_S consists of those edges in $E(G)$ both of whose endpoints lie in S. This induced subgraph will be denoted by G_S. For $k \geq 1$, a graph $G = (V, E)$ is called k**-connected** if for every $l < k$ and every W$=\{v_1, \dots, v_l\} \subseteq V$, the subgraph of G induced on $V \setminus W$ is connected. In particular, a 1-connected graph is a connected graph, and G is not 2-connected if and only if either G is not connected or there is some $v \in V$ such that $G_{V \setminus \{v\}}$ is not connected. A vertex v with the property just described is called a **cutpoint** of G. For $v \in V(G)$, the **neighborhood** $N_G(v)$ of v in G consists of all those vertices w such that $\{v, w\} \in E(G)$.

If $G = (V, E)$ is not k-connected, then the graph obtained from G by removing an edge $e \in E$ is not k-connected. Thus for fixed n and k, the collection

$$\Delta_n^k := \{G \in \Delta(n) : G \text{ is not } k - \textit{connected}\}$$

is a subcomplex of $\Delta(n)$. Define

$$\Delta^k(n) := \Delta(n)/\Delta_n^k$$

to be the quotient of the simplex $\Delta(n)$ (that is, the geometric realization) by the subcomplex Δ_n^k. The usual decomposition of $\Delta(n)$ into open simplices determines a cell decomposition of $\Delta^k(n)$ which makes $\Delta^k(n)$ a CW-complex. The cells in this decomposition include one point p_0 (corresponding to Δ_n^k) and, for $m \geq 0$, one m-cell G for each m-dimensional face G of $\Delta(n)$ such that G is k-connected. This decomposition, along with a chosen orientation on $\Delta(n)$, determines a cellular chain complex for $\Delta^k(n)$. The element of this complex which corresponds to a given k-connected graph G will be denoted by G. The reader may refer to [6] for the necessary definitions and theorems on cell complexes.

The complexes $\Delta^1(n)$ and $\Delta^2(n)$ are important in Vassiliev's study of knots in $\mathbf{R}^3$ (see [12, 13, 14]). In particular, the homology groups of these complexes appear as terms in spectral sequences used by Vassiliev to approximate the homology of the discriminant Σ in the space M of all smooth maps from S^1 to $\mathbf{R}^3$. Σ consists of all maps in M which are not embeddings, and the connected components of $\mathcal{M} \setminus \Sigma$ correspond to (equivalence classes of) knots. Thus a knot invariant (that is, a map from the set of equivalence classes of knots to $\mathbf{Z}$) is just an element of the cohomology group $H^0(\mathcal{M} \setminus \Sigma)$. Certain elements of the cohomology of $\mathcal{M} \setminus \Sigma$ can be determined from the homology of finite dimensional approximations to Σ, and the homology of $\Delta^1(n)$ and that of $\Delta^2(n)$ appear in spectral sequences which are used to determine the homology of these approximations. This process is described in the papers of Vassiliev cited above.

Since $\Delta(n)$ is contractible, determining the homology of $\Delta^k(n)$ is equivalent to determining the homology of Δ_n^k (see for example [7]). In fact, $\Delta^k(n)$ is homotopy equivalent to the suspension of Δ_n^k. Using standard methods from topological combinatorics, it can be shown that Δ_n^1 has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$, so

$$\widetilde{H}_i(\Delta^1(n)) \cong \begin{cases} \mathbf{Z}^{(n-1)!}, & i = n-2 \\ 0, & \text{otherwise.} \end{cases}$$

This result is not entirely satisfactory, as it is useful in Vassiliev's theory to have an explicit basis for $\widetilde{H}_{n-2}(\Delta^1(n))$. Such a basis was determined by Vassiliev in [13].

Theorem 1.1 ([13], Theorem 2) *Let $\mathcal{P}_n$ be the collection of all graphs G on vertex set $[n]$ whose edges form a path $1 = v_1, v_2, \ldots, v_n$. Then $\{G : G \in \mathcal{P}_n\}$ is a basis for $\widetilde{H}_{n-2}(\Delta^1(n))$.*

The homotopy type of Δ_n^2 has also been determined, in [1] and [10]. The result is as follows.

Theorem 1.2 ([1, 10]) *The complex Δ_n^2 has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $2n-5$. In particular,*

$$\widetilde{H}_i(\Delta^2(n)) \cong \begin{cases} \mathbf{Z}^{(n-2)!} & i = 2n-4, \\ 0 & \textit{otherwise.} \end{cases}$$

As above, this result is not entirely satisfactory for Vassiliev's theory. The purpose of this paper is to provide an explicit basis for $\widetilde{H}_{2n-4}(\Delta^2(n))$, thereby solving Problem 2 posed in [14].

In order to describe this basis, more definitions and notation will be needed. For $n \geq 3$, an ***n*-gon** is a graph on vertex set $[n]$ with exactly n edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, $\{v_n, v_1\}$, where $\{v_1, \ldots, v_n\} = [n]$. An n-gon can be drawn in the plane as a convex polygon with n sides. Let P be an n-gon. A **triangulation** of P is a graph T on $[n]$ which can be drawn in the plane as a triangulation of P, that is, $E(P) \subseteq E(T)$ and the $n-3$ edges of T which are not in $E(P)$ divide the interior of the polygon determined by P into $n-2$ triangles. These remaining edges do not intersect except at vertices of P. One can define a triangulation combinatorially as follows. A graph T on $[n]$ is a triangulation of an n-gon P if $E(P) \subseteq E(T)$, $|E(T) \setminus E(P)| = n-3$ and $E(T) \setminus E(P)$ does not contain edges $\{v_i, v_k\}$ and $\{v_j, v_l\}$ with $i < j < k < l$. If T is a triangulation of P, the edges of P will be called **external** edges of T and the remaining $n-3$ edges of T will be called **internal** edges of T. Each internal edge is contained in exactly two triangles while each external edge is contained in a unique triangle. The set of all triangulations of a fixed n-gon P will be denoted by $\mathcal{T}(P)$. Since P is 2-connected, so is each $T \in \mathcal{T}(P)$. Each $T \in \mathcal{T}(P)$ has $2n-3$ edges and therefore is a $(2n-4)$-cell in $\Delta^2(n)$. We can now state a version of our main theorem.

Theorem 1.3 *Let $\mathcal{P}_n$ be the set of all n-gons P such that $\{1,2\} \in E(P)$. Then for any $P \in \mathcal{P}_n$ there exists a function $f : \mathcal{T}(P) \to \{\pm 1\}$ such that*

$$c_P := \sum_{T \in \mathcal{T}(P)} f(T)T$$

represents a nontrivial element of $\widetilde{H}_{2n-4}(\Delta^2(n))$. The set $\{c_P : P \in \mathcal{P}_n\}$ is a basis for $\widetilde{H}_{2n-4}(\Delta^2(n))$.

The cellular homology $\widetilde{H}_*(\Delta^2(n))$ is isomorphic to the homology of the quotient complex of the simplicial chain complex of $\Delta(n)$ by that of Δ_n^2, and it is this homology that we will compute. The sign functions f are determined by the boundary map in the simplicial chain complex of $\Delta(n)$, which is in turn determined by a chosen orientation of $\Delta(n)$. A formula for $f(T)$ is given in section 4. Since the action of S_n on

$\Delta^2(n)$ by vertex permutation induces an action on $\widetilde{H}_{2n-4}(\Delta^2(n))$, the condition that $\{1,2\} \in E(P)$ is not necessary in order that c_P represent a nontrivial homology class, and understanding how to write such a c_P in terms of elements of the given basis is essential for understanding this homology representation of S_n. This representation was determined in [1], and appears in several other algebraic, topological and combinatorial settings (see [1] for a list of references and see [11] for other results).

The proof of Theorem 1.3 is essentially a topological version of a combinatorial approach used by Rodica Simion, who calculated the reduced Euler characteristic of Δ_n^2 when Theorem 1.2 was still a conjecture ([8]). I thank her for making this calculation available. I thank Svante Linusson and Michelle Wachs for many helpful comments, and I thank Robin Forman for explaining his results on discrete Morse theory during the special year on combinatorics at MSRI in 1996-97.

2 Forman's discrete Morse theory

The main tool in the proof of Theorem 1.3 is the discrete Morse theory of Forman, which is described in the paper [3]. The results from [3] which are needed here will be restated below in a different language from that of [3]. This language was introduced by Chari in [2]. It does not add anything to Forman's theory, but it simplifies the bookkeeping that arises when one attempts to define an explicit discrete Morse function on a given complex.

We begin by recalling the definition of the fundamental object of Forman's theory, the discrete Morse function. Let Σ be a finite CW-complex and let σ, τ be cells in Σ. Write $\sigma \leq \tau$ if $\sigma \subseteq \overline{\tau}$, and write $\sigma \ll \tau$ if $\sigma \leq \tau$ and $\dim \sigma = \dim \tau - 1$.

For a cell τ of Σ with $\dim \tau = m$, let $h_\tau : B^m \to \Sigma$ be the characteristic map for τ, so h_τ is a continuous function which induces a homeomorphism between the interior of the (closed) ball B^m and τ. For $\sigma \ll \tau$, call σ a **regular face** of τ if the two conditions

- $h_\tau : h_\tau^{-1}(\sigma) \to \sigma$ is a homeomorphism, and
- $\overline{h^{-1}(\sigma)}$ is homeomorphic to B^{m-1}

are satisfied.

Let $K(\Sigma)$ be the set of cells of Σ and let $f : K(\Sigma) \to \mathbf{R}$ be any function. For $\tau \in K(\Sigma)$ define

- $B_f^+(\tau) := \{\sigma \in K(\Sigma) : \tau \ll \sigma \text{ and } f(\tau) \geq f(\sigma)\}$, and
- $B_f^-(\tau) := \{\sigma \in K(\Sigma) : \sigma \ll \tau \text{ and } f(\sigma) \geq f(\tau)\}$.

Note that $\tau \in B_f^+(\sigma)$ if and only if $\sigma \in B_f^-(\tau)$.

A **discrete Morse function** on Σ is a function $f : K(\Sigma) \to \mathbf{R}$ which satisfies the three conditions

(F1) if $\sigma, \tau \in K(\Sigma)$ and $\sigma \in B_f^-(\tau)$ then σ is a regular face of τ,

(F2) $|B_f^+(\tau)| \leq 1$ for all $\tau \in K(\Sigma)$, and

(F3) $|B_f^-(\tau)| \leq 1$ for all $\tau \in K(\Sigma)$.

As noted in Lemma 2.5 of [3], if f is a discrete Morse function on Σ and $\tau \in K(\Sigma)$ then at least one of $B_f^+(\tau), B_f^-(\tau)$ is empty, so

(F*) $|B_f^+(\tau) \bigcup B_f^-(\tau)| \leq 1$.

Call $\tau \in K(\Sigma)$ an **f-critical m-cell** if $\dim \tau = m$ and $B_f^+(\tau) \bigcup B_f^-(\tau) = \emptyset$. The fundamental result of [3] is as follows.

Theorem 2.1 ([3, Corollary 3.5]) *Let Σ be a finite CW-complex and let $f : K(\Sigma) \to \mathbf{R}$ be a discrete Morse function. For $m \in \mathbf{N} \bigcup \{0\}$ let $c_m(f)$ be the number of f-critical m-cells of Σ. Then Σ is homotopy equivalent to a CW-complex which has exactly $c_m(f)$ cells of dimension m for each $m \in \mathbf{N} \bigcup \{0\}$.*

As we will see shortly, the paper [3] contains many other interesting results, some of which can be used to determine the homology of Σ, given a good enough Morse function f on Σ. We will restate the results which we will need in the language of [2]. Our goal is to apply Forman's theory to the complexes $\Delta^2(n)$, and we will eventually state the necessary results for CW-complexes of the form Σ/Γ, where Σ is a simplicial complex and Γ is a nonempty subcomplex. To give the reader a gentle introduction to the new language, and for possible future reference, we will first introduce Chari's language in the case where the CW-complex Σ is a simplicial complex, whose cells are just the nonempty faces of Σ. Note that in this case, if $\sigma, \tau \in K(\Sigma)$ and $\sigma \ll \tau$ then Σ is a regular face of τ. Thus condition (F1) is satisfied by every function $f : K(\Sigma) \to \mathbf{R}$.

So, let Σ be a finite simplicial complex. Define the **face digraph** $D(\Sigma) = (V, A)$ to be the directed graph whose vertex set $V = V(\Sigma)$ contains one vertex (also denoted by σ) for each nonempty face σ of Σ, with an arc (that is, a directed edge) (τ, σ) from τ to σ in $A = A(\Sigma)$ if and only if $\sigma \ll \tau$. In other words, $D(\Sigma)$ is the directed graph obtained from the Hasse diagram of the face poset of Σ (see [9], p. 98) by directing each edge downwards. The vertices τ and σ will be referred to as **endpoints** of the arc (τ, σ).

For $M \subseteq A(\Sigma)$, define

$$M^{op} := \{(\sigma, \tau) : (\tau, \sigma) \in M\},$$

and define the digraph $D_M(\Sigma)$ by

$$D_M(\Sigma) := (V(\Sigma), (A(\Sigma) \setminus M) \cup M^{op}),$$

so $D_M(\Sigma)$ is the digraph obtained from $D(\Sigma)$ by reversing the directions of the arcs in M.

A **Morse matching** on $D(\Sigma)$ is a collection of arcs $M \subseteq A(\Sigma)$ which satisfies the two conditions

(S1) each $\sigma \in V(\Sigma)$ is an endpoint of at most one arc in M, and

(S2) the digraph $D_M(\Sigma)$ contains no directed cycle.

Recall that a **directed cycle** in a digraph D is simply a sequence

$$(v_1, v_2), (v_2, v_3), \ldots, (v_{r-1}, v_r = v_1)$$

of $r - 1 > 2$ arcs in D such that $v_1, v_2, \ldots v_{r-1}$ are all distinct. Such a cycle will often be denoted by $v_1, v_2, \ldots, v_r = v_1$ in what follows.

Morse matchings on $D(\Sigma)$ correspond to classes of Morse functions on Σ as follows. Let f be a discrete Morse function on Σ. Define

$$M_f := \left\{(\tau, \sigma) \in A(\Sigma) : \sigma \in B_f^-(\tau)\right\}.$$

Now the fact that each $\tau \in K(\Sigma)$ satisfies condition (F*) guarantees that M_f satisfies condition (M1). Assume for contradiction that $D_{M_f}(\Sigma)$ contains a directed cycle $\sigma_1, \sigma_2, \ldots, \sigma_r = \sigma_1$. If $(\sigma_{i+1}, \sigma_i) \notin M_f$ then $\sigma_{i+1} \ll \sigma_i$ and $f(\sigma_i) > f(\sigma_{i+1})$. If $(\sigma_{i+1}, \sigma_i) \in M_f$ then $\sigma_i \ll \sigma_{i+1}$ and $f(\sigma_i) \geq f(\sigma_{i+1})$. Therefore, $f(\sigma_i) \geq f(\sigma_{i+1})$ for all i, and since $\sigma_r = \sigma_1$ we have $f(\sigma_i) = f(\sigma_1)$ for all i. This forces $(\sigma_{i+1}, \sigma_i) \in M_f$ for all i, which is impossible since $\sigma_r = \sigma_1$. Therefore, M_f satisfies condition (S2) and is a Morse matching on $D(\Sigma)$.

Conversely, let M be a Morse matching on $D(\Sigma)$. The fact that M satisfies (S2) guarantees that there exists some function $f = f_M : K(\Sigma) \to \mathbf{R}$ such that $f(\tau) > f(\sigma)$ whenever (τ, σ) is an arc of $D_M(\Sigma)$. For any such function $f = f_M$ and any $\tau \in K(\Sigma)$ we have

- $B_f^+(\tau) = \{\sigma \in K(\Sigma) : (\sigma, \tau) \in M\}$, and
- $B_f^-(\tau) = \{\sigma \in K(\Sigma) : (\tau, \sigma) \in M\}$.

The fact that M satisfies (S1) guarantees that f satisfies (F2) and (F3), so f is a discrete Morse function on Σ. It is straightforward to show that $M_f = M$. The relation $f \sim g$ if and only if $M_f = M_g$ is an equivalence relation on the set of discrete Morse functions on Σ, and the map $[f] \mapsto M_f$ is a bijection between the set of $\sim$-equivalence classes and the set of Morse matchings on $D(\Sigma)$.

Now we define Morse matchings for CW-complexes of the form Σ/Γ, where Σ is a simplicial complex and Γ is a nonempty subcomplex. Such a complex inherits a cell decomposition from the decomposition of Σ into its open faces. Thus Σ/Γ contains one m-cell (also called σ) for each m-cell $\sigma \in K(\Sigma) \setminus K(\Gamma)$ and one additional 0-cell p_0 (which corresponds to the subcomplex Γ). Let $\pi : \Sigma \to \Sigma/\Gamma$ be the standard quotient map. If $\tau \in K(\Sigma) \setminus K(\Gamma)$ with $\dim \tau = m$ and $h_\tau : B^m \to \Sigma$ is the characteristic map for τ in Σ, then $\pi h_\tau : B^m \to \Sigma/\Gamma$ is the characteristic map for τ in Σ/Γ. It follows that if $\sigma \in K(\Sigma) \setminus K(\Gamma)$ and $\sigma \ll \tau$ in Σ then σ is a regular face of τ in Σ/Γ. Therefore, any function $f : K(\Sigma/\Gamma) \to \mathbf{R}$ such that $f(p_0) < f(\tau)$ for all

$\tau \in K(\Sigma) \setminus K(\Gamma)$ satisfies conditon (F1). Morse matchings for Σ/Γ will be defined by ignoring p_0 and working with the elements of $K(\Sigma) \setminus K(\Gamma)$.

So, for a simplicial complex Σ and a nonempty subcomplex $\Gamma \subseteq \Sigma$, define the **relative face digraph** $D(\Sigma, \Gamma)$ to be the subdigraph of $D(\Sigma)$ induced on the vertex set $V(\Sigma, \Gamma) \subseteq V(\Sigma)$ consisting of all vertices corresponding to faces of Σ which are *not* in Γ. Denote the set of arcs of $D(\Sigma, \Gamma)$ by $A(\Sigma, \Gamma)$. Also, denote the set of faces of Σ which are not faces of Γ by $\Sigma \setminus \Gamma$ (instead of $K(\Sigma) \setminus K(\Gamma)$ as was done previously). So, $A(\Sigma, \Gamma)$ consists of all arcs (τ, σ) such that $\tau, \sigma \in \Sigma \setminus \Gamma$ and $(\tau, \sigma) \in A(\Sigma)$. Note that there is no element of $V(\Sigma, \Gamma)$ which corresponds to the 0-cell $p_0 \in K(\Sigma/\Gamma)$. As we did for somplicial complexes, for $M \subseteq A(\Sigma, \Gamma)$ we define

$$M^{op} := \{(\sigma, \tau) : (\tau, \sigma) \in M\},$$

and define the digraph $D_M(\Sigma)$ by

$$D_M(\Sigma, \Gamma) := (V(\Sigma, \Gamma), (A(\Sigma, \Gamma) \setminus M) \cup M^{op}),$$

so $D_M(\Sigma, \Gamma)$ is the digraph obtained from $D(\Sigma, \Gamma)$ by reversing the directions of the arcs in M.

A **Morse matching** on $D(\Sigma, \Gamma)$ is a collection of arcs $M \subseteq A(\Sigma, \Gamma)$ which satisfies the two conditions

(M1) each $\sigma \in V(\Sigma, \Gamma)$ is an endpoint of at most one arc in M, and

(M2) the digraph $D_M(\Sigma, \Gamma)$ contains no directed cycle.

Arguing as we did above in the case of simplicial complexes, one can show that Morse matchings on $D(\Sigma, \Gamma)$ correspond to equivalence classes of discrete Morse functions on the quotient complex Σ/Γ for which p_0 is a critical cell. This correspondence allows us to restate Forman's results in terms of Morse matchings (for quotients of simplicial complexes by nonempty subcomplexes). Before doing so, we make two remarks.

- Forman's theory for arbitrary CW-complexes can be restated in Chari's language. In fact, something similar to such a restatement is given in section 9 of [3].
- It is straightforward to show that if Σ is a simplicial complex and M is a Morse matching on $D(\Sigma)$ then there is at least one 0-cell $v \in K(\Sigma)$ such that no arc in M has v as an endpoint. Therefore, M is also a Morse matching on $D(\Sigma, \{v\})$. Now Σ and $\Sigma/\{v\}$ are identical as CW-complexes. Thus Morse matchings for simplicial complexes are special cases of Morse matchings for quotient complexes.

We are now ready to state the necessary results from Forman's theory in Chari's language. Let M be a Morse matching on $D(\Sigma, \Gamma)$. An **M-critical m-cell** is an m-dimensional face in $\Sigma \setminus \Gamma$ whose corresponding vertex in $D(\Sigma, \Gamma)$ is not an endpoint of any arc in M. Given the Morse matching M, we define a bijection, also called M, on $V(\Sigma, \Gamma)$ as follows.

1. If σ is M-critical then $M(\sigma) = \sigma$.

2. If either of (σ, τ) or (τ, σ) is in M, then $M(\sigma) = \tau$ and $M(\tau) = \sigma$.

Condition (M1) guarantees that M is a well-defined bijection. As noted in Theorem 2.1, the basic result of Forman's theory (for quotients of simplicial complexes) is the following theorem.

Theorem 2.2 ([3, Corollary 3.5]) *Let Σ be a finite simplicial complex and let Γ be a nonempty subcomplex. Let M be a Morse matching on $D(\Sigma, \Gamma)$. Then Σ/Γ is homotopy equivalent to a CW-complex which has one point p_0 and one additional m-cell for each M-critical m-cell.*

As noted above, the 0-cell p_0 corresponds to the subcomplex Γ, which we can imagine as being shrunk to a point in order to obtain the quotient complex Σ/Γ from the original complex Σ. A particular case of Theorem 2.2 is that if M is a Morse matching on $D(\Sigma, \Gamma)$ with r M-critical m-cells and no other critical cells, then Σ/Γ has the homotopy type of a wedge of r spheres of dimension m. The first step in the proof of Theorem 1.3 is to produce a Morse matching M_n on $D(\Delta(n), \Delta_n^2)$ whose critical cells are $(n-2)!$ cells of dimension $2n-4$ (that is, 2-connected graphs with $2n-3$ edges), thereby reproving Theorem 1.2. Then further results from [3] will be invoked to find the desired basis. These results are described below.

Let $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Gamma)$ be, respectively, the simplicial chain complexes for Σ and Γ with respect to some orientation of Σ, and let

$$\mathcal{C}(\Sigma/\Gamma) : C_l(\Sigma)/C_l(\Gamma) \stackrel{\partial_l}{\to} C_{l-1}(\Sigma)/C_{l-1}(\Gamma) \stackrel{\partial_{l-1}}{\to} \ldots \stackrel{\partial_1}{\to} C_0(\Sigma)/C_0(\Gamma) \to 0$$

be the quotient complex (see Section 9 of [7]). Then each $C_i := C_i(\Sigma)/C_i(\Gamma)$ has a basis indexed by the i-dimensional faces in $\Sigma \setminus \Gamma$. The element of this basis corresponding to $\sigma \in \Sigma \setminus \Gamma$ will also be denoted by σ. Define a scalar product $\langle ., . \rangle$ on C_i so that this basis is orthonormal. Of course the (reduced) homology we want to find, namely that of the cell complex $\Delta^2(n)$, is isomorphic to the homology of the quotient complex $\mathcal{C}(\Delta(n)/\Delta_n^2)$.

Let M be a Morse matching on $D(\Sigma, \Gamma)$. For a basis element $\sigma \in C_i$, define

$$\mathcal{V}(\sigma) = \mathcal{V}_M(\sigma) := \begin{cases} -\langle \partial_{i+1} M(\sigma), \sigma \rangle M(\sigma), & \text{if } \sigma \text{ is a face of } M(\sigma) \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathcal{V}(\sigma) \in \{\pm\tau\}$ if $(\tau, \sigma) \in M$, and $\mathcal{V}(\sigma) = 0$ if σ is M-critical or if $(\sigma, \tau) \in M$ for some τ. Now for each i, extend $\mathcal{V}$ to a linear map from C_i to C_{i+1}. Then both $\partial_{i+1}\mathcal{V}$ and $\mathcal{V}\partial_i$ are linear maps on C_i, and the **discrete gradient flow** $\Phi = \Phi(M) : C_i \to C_i$ is defined for each i by

$$\Phi := 1 + \partial_{i+1}\mathcal{V} + \mathcal{V}\partial_i,$$

where $1 : C_i \to C_i$ is the identity map.

Theorem 2.3 ([3, Theorem 6.4]) *For each $i \in [l]$,*

$$\Phi\partial_i = \partial_i\Phi.$$

For $i \in [l]$ define

$$C_i^{\Phi} := \{c \in C_i : \Phi(c) = c\}.$$

By Theorem 2.3, ∂_i maps C_i^{Φ} to C_{i-1}^{Φ}. The **Morse complex** for M is the complex

$$\mathcal{C}^{\Phi} : C_l^{\Phi} \xrightarrow{\partial_l} C_{l-1}^{\Phi} \xrightarrow{\partial_{l-1}} \ldots \xrightarrow{\partial_1} C_0^{\Phi} \to 0.$$

Forman shows that the homology of the Morse complex is isomorphic to that of the original complex. His first step is to show that repeated application of Φ to some $c \in C_i$ eventually produces a Φ-invariant chain.

Theorem 2.4 ([3, Theorem 7.2]) *There is some $N = N(M) \in \mathbf{N}$ such that $\Phi^N = \Phi^{N+1}$ on each C_i.*

Thus for any basis element $\sigma \in C_i$, we have $\Phi^N(\sigma) \in C_i^{\Phi}$. Since Φ commutes with ∂_i, so does Φ^N, so $\Phi^N : \mathcal{C} \to \mathcal{C}^{\Phi}$ is a chain map. Let $\iota : \mathcal{C}^{\Phi} \to \mathcal{C}$ be the inclusion map. The maps Φ^N and ι induce homomorphisms of (reduced) homology $\Phi_*^N : \widetilde{H}_*(\mathcal{C}) \to \widetilde{H}_*(\mathcal{C}^{\Phi})$ and $\iota_* : \widetilde{H}_*(\mathcal{C}^{\Phi}) \to \widetilde{H}_*(\mathcal{C})$.

Theorem 2.5 ([3, Theorem 7.3]) *The maps Φ_*^N and ι_* are inverse to each other. In particular, $\widetilde{H}_*(\mathcal{C}(\Sigma/\Gamma)) \cong \widetilde{H}_*(\mathcal{C}^{\Phi})$.*

Thus if $C_{i-1}^{\Phi} = C_{i+1}^{\Phi} = 0$ then a basis for C_i^{Φ} is also a basis for $\widetilde{H}_i(\Sigma/\Gamma)$. The Morse matching M_n on $D(\Delta(n), \Delta_n^2)$ will satisfy $C_i^{\Phi} = 0$ for all $i \neq 2n-4$. To prove this, further results of Forman will be used. Let M be a Morse matching on $D(\Sigma, \Gamma)$. For $m \in \mathbf{N}$ let $C_m(M)$ be the span in C_m of the M-critical m-cells, that is,

$$C_m(M) := \left\{\sum a_\sigma \sigma \in C_m : a_\sigma = 0 \text{ if } M(\sigma) \neq \sigma\right\}.$$

Theorem 2.6 ([3, Theorem 8.2]) *The restriction of Φ^N to $C_m(M)$ is an isomorphism between $C_m(M)$ and C_m^{Φ}.*

In particular, $\left\{\Phi^N(\sigma) : \sigma \text{ is an } M\text{-critical } m\text{-cell}\right\}$ is a basis for C_m^{Φ}. This basis will be determined for each M_n, up to signs which depend on the chosen orientation of $\Delta(n)$.

3 The Morse matchings M_n

In this section and the next, diagrams indicating the main ideas in some of the definitions and proofs will be provided in order to make clearer some straightforward concepts which might be rendered confusing by the large amount of necessary notation. The diagrams are provided after the ideas have been introduced and discussed.

To define the Morse matchings M_n, some additional notation and definitions will be necessary. As above, a face of $\Delta(n)$ corresponding to a graph G will be denoted by G. For a graph G and an edge $e = \{x, y\} \in E(G)$, the graph obtained from G by removing e will be denoted by $G - e$ or by $G - \{x, y\}$. Similarly, if $e \notin E(G)$ then the graph obtained from G by adding e will be denoted by $G + e$ or by $G + \{x, y\}$. The digraph $D(\Delta(n), \Delta_n^2)$ will be denoted by $D(n) = (V(n), A(n))$, and for any $M \subseteq A(n)$, the digraph $D_M(\Delta(n), \Delta_n^2)$ will be denoted by $D_M(n)$. By definition, the elements of $V(n)$ correspond to the 2-connected graphs on vertex set $[n]$. Our goal is to produce a Morse matching M on $D(n)$ with $(n-2)!$ M_n-critical $(2n-4)$-cells and no other critical cells. Assume from now on that $n \geq 3$.

Say $M \subseteq A(n)$ is a matching, that is, M satisfies (M1). If M does not satisfy (M2) there is some directed cycle $\mathcal{D} = v_1, \ldots, v_{r-1}, v_r = v_1$ in $D_M(n)$. Fix $i \in [r-1]$ and let $H = v_i$. Then $\mathcal{D}$ contains exactly two arcs (G, H) and (H, I) which have H as an endpoint. It cannot be the case that both of these arcs are in M, since this would violate (M1). Thus either $H = G - e_1$ and $I = H + e_2$ with $(H, I) \in M^{op}$ or $H = G + e_1$ and $I = H - e_2$ with (G, H) in M^{op}. This argument can be applied at each v_i to obtain the following result.

Proposition 3.1 *Let $M \subseteq A(n)$ satisfy (M1). If M is not a Morse matching then there exists a directed cycle*

$$G_1, H_1, G_2, H_2, \ldots, G_{r-1}, H_{r-1}, G_r = G_1$$

in $D_M(n)$ satisfying the following three conditions.

1. *For each $i \in [r-1]$ there exist edges $a_i, b_i \in E(H_i)$ such that $G_i + a_i = H_i = G_{i+1} + b_i$.*
2. *For each $i \in [r-1]$, we have $(H_i, G_i) \in M$.*
3. *The multisets $\{a_i : i \in [r-1]\}$ and $\{b_i : i \in [r-1]\}$ are equal.*

Moreover, every directed cycle in $D_M(n)$ satisfies these conditions.

The third condition holds since the graph obtained from G_1 by adding all the a_i and removing all the b_i is again G_1.

If $G = (V, E)$ is a finite connected graph then there is a unique decomposition $V = \bigcup_{i=1}^{r} B_i$ such that each G_{B_i} is a maximal 2-connected induced subgraph of G (see for example [5]). For $1 \leq i < j \leq r$, $|B_i \cap B_j| \leq 1$ and $c \in V$ is a cutpoint of G if and only if there exist distinct i, j such that $c \in B_i \cap B_j$. The B_i are called the

blocks of G. Define B_G to be the bipartite graph with one vertex X_i for each block B_i of G, one vertex Y_j for each cutpoint c_j of G, and an edge $\{X_i, Y_j\}$ if and only if $c_j \in B_i$. Then B_G is a tree (that is, B_G is connected and has no cycles).

Proposition 3.2 *Let $G = (V, E)$ be a 2-connected finite graph with $|V| > 2$ and let $e = \{v, w\} \in E$. Let $H = G - e$. Then H is connected. Assume H is not 2-connected. Let $B_1, \ldots, B_r$ be the blocks of H and let $c_1, \ldots, c_s$ be the cutpoints of H. Then $s = r - 1$ and the indices may be chosen so that B_H is the path $X_1, Y_1, X_2, Y_2, \ldots, X_{r-1}, Y_{r-1}, X_r$ with $v \in B_1 \setminus \{c_1\}$ and $w \in B_r \setminus \{c_{r-1}\}$.*

Proof: If H is not connected then, since $V \neq \{v, w\}$, one of v, w is a cutpoint of G, a contradiction. Thus H is connected and B_H is a tree. Assume for contradiction that the second claim of the proposition is false. Then there is some X_i of degree one in B_H such that $B_i \cap \{v, w\} = \emptyset$. Let $\{X_i, Y_j\}$ be the unique edge in B_H containing X_i. Then there are no edges between $B_i \setminus c_j$ and $V \setminus B_i$ in H. Since neither endpoint of e is in B_i, the same condition holds in G. Thus c_j is a cutpoint of G, which gives the desired contradiction. □

The Morse matchings M_n are defined recursively. Note that $V(3)$ contains only the complete graph K_3 (a 2-dimensional face). We define M_3 to be the empty matching, so that K_3 is M_3-critical. Now assume that $n > 3$ and the Morse matching M_{n-1} has been defined. The Morse matching M_n on $D(n)$ will be the disjoint union of Morse matchings $M_n^{(i)}$ for $1 \leq i \leq 3$.

If $G \in V(n)$ and $\{1,2\} \notin E(G)$ then $G + \{1,2\} \in V(n)$. Set

$$M_n^{(1)} := \{(G + \{1,2\}, G) : \{1,2\} \notin E(G)\}.$$

Lemma 3.3 *1. $M_n^{(1)}$ is a Morse matching on $D(n)$.*

2. The $M_n^{(1)}$-critical cells of $D(n)$ are those G such that G is 2-connected, $\{1,2\} \in E(G)$ and $G - \{1,2\}$ is not 2-connected.

Proof: Fix $G \in V(n)$. If $\{1,2\} \notin E(G)$ then $(G + \{1,2\}, G)$ is the only arc in $M_n^{(1)}$ with G as an endpoint. If $\{1,2\} \in E(G)$ and $G - \{1,2\} \in V(n)$ then $(G, G - \{1,2\})$ is the only arc in $M_n^{(1)}$ with G as an endpoint. If $\{1,2\} \in E(G)$ and $G - \{1,2\} \notin \Delta^2(n)$ then G is not an endpoint of any arc in $M_n^{(1)}$. Thus (M1) is satisfied, and if (M2) is satisfied then $M_n^{(1)}$ is a Morse matching whose critical cells are as described in part (2) of the lemma. Assume for contradiction that (M2) is not satisfied. Let

$$G_1, H_1, G_2, H_2, \ldots, G_{r-1}, H_{r-1}, G_r = G_1$$

be a directed cycle in $D_{M_n^{(1)}}(n)$ satisfying the conditions given in Proposition 3.1. Since every arc in $M_n^{(1)}$ is of the form $(G + \{1,2\}, G)$, each $a_i = \{1,2\}$. Since $\{b_i\} = \{a_i\}$, each $b_i = \{1,2\}$. Thus $G_2 = H_1 - \{1,2\} = G_1$, giving the desired contradiction. □

Let $C_n^{(1)}$ be the set of $M_n^{(1)}$-critical cells of $D(n)$. For $G \in C_n^{(1)}$, define $K = K(G) := G - \{1,2\}$. By Proposition 3.2, K has blocks $B_1, \ldots, B_r$ such that $1 \in B_1$,

$2 \in B_r$ and $B_i \cap B_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Define $b(G)$ to be the unique element of $B_1 \cap B_2$. Note that if $b(G) \in \{1,2\}$ then $b(G)$ is a cutpoint of G, contradicting the fact that G is 2-connected. Thus $b(G) \notin \{1,2\}$. If $\{2, b(G)\} \notin E(G)$ then the blocks of $K + \{2, b(G)\}$ are B_1 and $\bigcup_{2 \leq i \leq r} B_i$, so $G + \{2, b(G)\} \in C_n^{(1)}$. Define

$$M_n^{(2)} := \left\{ (G + \{2, b(G)\}, G) : G \in C_n^{(1)}, \{2, b(G)\} \notin E(G) \right\}.$$

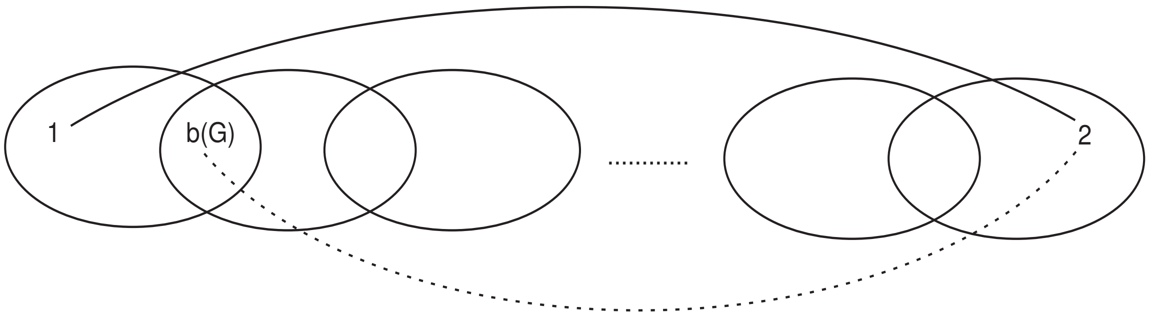


Figure 1: An element $G \in C_n^{(1)}$. The ellipses represent the blocks of $G - \{1,2\}$. The partner of G in $M_n^{(2)}$ is obtained by adding or removing (if possible) the edge $\{2, b(G)\}$.

Lemma 3.4 *1. $M_n^{(1)} \cup M_n^{(2)}$ is a Morse matching on $D(n)$.*

2. The $(M_n^{(1)} \cup M_n^{(2)})$-critical cells of $D(n)$ are those $G \in C_n^{(1)}$ such that $N_G(2) = \{1, b(G)\}$.

Proof: Since the endpoints of all arcs in $M_n^{(2)}$ are $M_n^{(1)}$-critical cells, no vertex of $D_{M_n^{(1)} \cup M_n^{(2)}}(n)$ is an endpoint of both an arc in $M_n^{(1)}$ and an arc in $M_n^{(2)}$. Since $M_n^{(1)}$ satisfies (M1) by Lemma 3.3, in order to show that $M_n^{(1)} \cup M_n^{(2)}$ satisfies (M1) it suffices to show that no $G \in C_n^{(1)}$ is an endpoint of two arcs in $M_n^{(2)}$.

Fix $G \in C_n^{(1)}$ and let $B_1, \ldots, B_r$ be the blocks of $K = K(G)$. Say $\{2, b(G)\} \notin E(G)$. We know that $(G + \{2, b(G)\}, G)$ is an arc in $M_n^{(2)}$. Assume for contradiction that there is another arc in $M_n^{(2)}$ which has G as an endpoint. This arc must be of the form $(G, H = G - e)$ with $e = \{2, b(H)\}$. However, $b(H) \in B_1$ and there is no edge $\{2, x\} \in E(G)$ such that $x \in B_1$, and we have the desired contradiction. Thus $(G + \{2, b(G)\}, G)$ is the unique arc in $M_n^{(2)}$ which has G as an endpoint,

Say $\{2, b(G)\} \in E(G)$. Then $K = K(G)$ has exactly two blocks B_1, B_2 with $1 \in B_1$, $2 \in B_2$ and $B_1 \cap B_2 = \{b(G)\}$. First assume $|B_2| > 2$. Then by Proposition 3.2 either $K_{B_2} - \{2, b(G)\}$ is 2-connected or $K_{B_2} - \{2, b(G)\}$ has blocks $W_1, \ldots, W_s$ with $b(G) \in W_1$, $2 \in W_s$ and $W_i \cap W_j \neq \emptyset$ if and only if $|i-j| \leq 1$. In the second case, $K(G - \{2, b(G)\})$ has blocks $B_1, W_1, \ldots, W_s$ and in either case $b(K(G - \{2, b(G)\})) = b(G)$. Thus $(G, G - \{2, b(G)\}) \in M_n^{(2)}$. Conversely, since $\{2, b(G)\} \in E(G)$, any arc in $M_n^{(2)}$ which has G as an endpoint is of the form (G, H) with $G = H + \{2, b(H)\}$. Since $\{2, b(H)\}$ does not have both endpoints in B_1, B_1 is a block in $K(H)$. The only element of B_1 which can be contained in two blocks of $K(H)$ is $b(G)$. Thus $b(H) = b(G)$ and $H = G - \{2, b(G)\}$, so $(G, G - \{2, b(G)\})$ is the unique arc in $M_n^{(2)}$ with G as an endpoint.

Now assume $|B_2| = 2$. Note that this condition holds if and only if $N_G(2) = \{1, b(G)\}$. Since every arc in $M_n^{(2)}$ is of the form $(H, H - \{2, x\})$ and there is no x such that $G - \{2, x\}$ is 2-connected, there is no arc of the form (G, H) in $M_n^{(2)}$. Since $\{2, b(G)\} \in E(G)$, there is no arc of the form (H, G) in $M_n^{(2)}$. Thus G is not an endpoint of any arc in $M_n^{(2)}$. We have now seen that (M1) is satisfied and that if (M2) is satisfied then $M_n^{(1)} \cup M_n^{(2)}$ is a Morse matching whose critical cells are described in (2) of the lemma.

Assume for contradiction that $D_{M_n^{(1)} \cup M_n^{(2)}}(n)$ contains a directed cycle

$$G_1, H_1, G_2, H_2, \ldots, G_{r-1}, H_{r-1}, G_r = G_1,$$

which must be as described in Proposition 3.1. First note that if $G \in C_n^{(1)}$ and $e \in E(G)$ then either $G - e \in C_n^{(1)}$ or $G - e$ is not 2-connected. Thus there is no arc (G, H) in $D(n)$ with $H \notin C_n^{(1)}$. There is also no such arc in $(M_n^{(1)} \cup M_n^{(2)})^{op}$, since arcs in $M_n^{(1)}$ have neither endpoint in $C_n^{(1)}$ and arcs in $M_n^{(2)}$ have both endpoints in $C_n^{(1)}$. Therefore, there is no arc $(G, H) \in D_{M_n^{(1)} \cup M_n^{(2)}}(n)$ with $G \in C_n^{(1)}$ and $H \notin C_n^{(1)}$. It follows that either all of the G_i and all of the H_i are in $C_n^{(1)}$ or none of them are. By Lemma 3.3, all of the G_i and all of the H_i are in $C_n^{(1)}$. Then each $a_i = \{2, b(G_i)\}$. It was shown in the paragraph before the previous one that $b(G_i) = b(H_i)$. Since 2 is an endpoint of each b_i, the block of G_{i+1} containing vertex 1 is the same as the block of H_i containing that vertex. Thus $b(G_{i+1}) = b(H_i)$. It follows that $G_2 = H_1 - \{2, b(H_1)\} = G_1$, giving the desired contradiction. □

Let $C_n^{(2)}$ be the set of $(M_n^{(1)} \cup M_n^{(2)})$-critical cells in $D(n)$. For $3 \leq i \leq n$ define

$$C_n^{(2)}(i) := \left\{ G \in C_n^{(2)} : b(G) = i \right\},$$

so $C_n^{(2)}$ is the disjoint union of the subsets $C_n^{(2)}(i)$. Each $C_n^{(2)}(i)$ consists exactly of those graphs G such that $N_G(2) = \{1, i\}$ and $G_{V \setminus \{2\}}$ is 2-connected. For each pair (i, n) of integers with $2 < i \leq n$ define the map $\phi_{i,n} : [n] \setminus \{2\} \to [n-1]$ by

$$\phi_{i,n}(j) := \begin{cases} j, & j < i \\ 2, & j = i \\ j - 1, & j > i. \end{cases}$$

Then each $\phi_{i,n}$ is a bijection which induces a map from the set of all graphs on vertex set $[n] \setminus \{2\}$ to the set of all graphs on vertex set $[n-1]$. This induced map will also be denoted by $\phi_{i,n}$. Define the function $\tau_{i,n} : \Delta(n) \to \Delta(n-1)$ by

$$\tau_{i,n}(G) := \phi_{i,n}(G_{[n] \setminus \{2\}}).$$

For each i, the map $\tau_{i,n}$ which sends induces an isomorphism between the subdigraph of $D(n)$ induced on $C_n^{(2)}(i)$ and $D(n-1)$. For $3 \leq i \leq n$ define

$$M_n^{(3)}(i) := \left\{ (\tau_{i,n}^{-1}(G), \tau_{i,n}^{-1}(H)) : (G, H) \in M_{n-1} \right\},$$

and

$$M_n^{(3)} := \bigcup_{i=3}^{n} M_n^{(3)}(i).$$

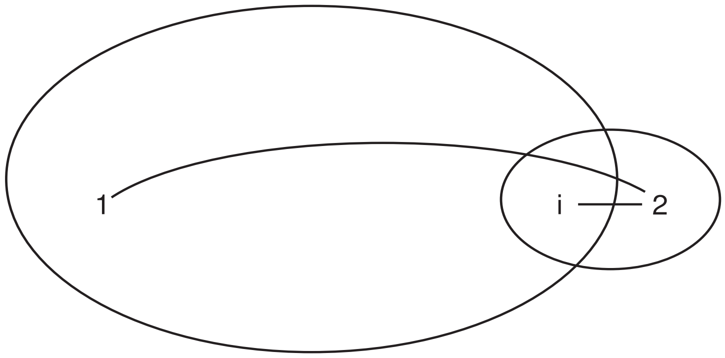


Figure 2: An element $G \in C_n^{(2)}$. The partner of G in $M_n^{(3)}$ is determined by the partner of $\tau_{i,n}(G)$ in M_{n-1}.

Finally, define

$$M_n := M_n^{(1)} \cup M_n^{(2)} \cup M_n^{(3)}.$$

Theorem 3.5 *1. M_n is a Morse matching on $D(n)$.*

2. The M_n-critical cells of $D(n)$ are those G satisfying the two conditions

- *$N_G(1) = \{2, \ldots, n\}$, and*
- *the subgraph of G induced on $[n] \setminus \{1\}$ is a path $2 = v_2, v_3, \ldots, v_n$.*

Proof: Both claims (1) and (2) hold when $n = 3$, and the proof goes by induction on n. Assume $n > 3$. Note that the endpoints of all arcs in $M_n^{(3)}$ lie in $C_n^{(2)}$ and that $M_n^{(1)} \cup M_n^{(2)}$ satisfies (M1) by Lemma 3.4. Thus if $M_n^{(3)}$ satisfies (M1) then so does M_n. It follows from the inductive hypothesis that each $M_n^{(3)}(i)$ satisfies (M1), and since the $C_n^{(2)}(i)$ are disjoint, $M_n^{(3)}$ satisfies (M1).

Assume for contradiction that $G_1, H_1, G_2, H_2, \ldots, G_{r-1}, H_{r-1}, G_r = G_1$ is a directed cycle in $D_{M_n}(n)$, as described in Proposition 3.1. Let

$$\chi = \{(H_i, G_i) : i \in [r-1]\},$$

so $\chi \subseteq M_n$. By Lemmas 3.3 and 3.4, $\chi \not\subseteq M_n^{(1)} \cup M_n^{(2)}$. Say $\chi \subseteq M_n^{(3)}$. Then since vertex 2 is not an endpoint of any a_i, it is not an endpoint of any b_i. Thus there is some $b \in \{3, \ldots, n\}$ such that $N_{G_i}(2) = \{1, b\}$ for all $i \in [r]$. This means that $\chi \subseteq M_n^{(3)}(b)$, which contradicts the inductive hypothesis. Therefore, χ is not contained in $M_n^{(3)}$ and $\{G_i\} \cup \{H_i\}$ includes some graphs in $C_n^{(2)}$ and some not in $C_n^{(2)}$.

If there is some i such that $G_i \notin C_n^{(1)}$ then $a_i = \{1,2\}$, so there is some j such that $b_j = \{1,2\}$. We may assume $j < r$. Then by the definition of $M_n^{(1)}$ we have $a_{j+1} = \{1,2\}$, so $G_{j+1} = G_j$, a contradiction. Thus $\{G_i\} \cup \{H_i\} \subseteq C_n^{(1)}$ and $\chi \subseteq M_n^{(2)} \cup M_n^{(3)}$. Note that for any $i \in [r-1]$ we have $G_i \in C_n^{(2)}$ if and only if $H_i \in C_n^{(2)}$. We may now assume that $H_1 \in C_n^{(2)}$ and $G_2 \in C_n^{(1)} \setminus C_n^{(2)}$.

For any graph X on $[n]$, denote by X^2 the subgraph of X induced on $[n] \setminus \{2\}$. Since $H_1 \in C_n^{(2)}$, H_1^2 is 2-connected and $N_{H_1}(2) = \{1, i\}$ for some $i \in \{3, \ldots, n\}$. Since G_2 is 2-connected, 2 is not an endpoint of b_2, so $N_{G_2}(2) = \{1, i\}$. Since $G_2 \notin C_n^{(2)}$, G_2^2 is not 2-connected. Now $(H_2, G_2) \in M_n^{(2)}$, so $a_2 = \{2, j\}$ for some $j \in \{3, \ldots, n\}$ and H_2^2 is not 2-connected. Thus G_3^2 is not 2-connected. It follows that $G_3 \in C_n^{(1)} \setminus C_n^{(2)}$, and that $H_3 \in C_n^{(1)} \setminus C_n^{(2)}$. We can now apply this last argument iteratively to show that $G_i \in C_n^{(1)} \setminus C_n^{(2)}$ for all $i > 2$. However, $G_r = G_1 \in C_n^{(2)}$, giving the desired contradiction under the assumption of the existence of a directed cycle. This completes the proof of the first claim.

The M_n-critical cells of $D(n)$ are those of the form $\tau_{i,n}^{-1}(G)$ where $i \in \{3, \ldots, n\}$ and G is an M_{n-1}-critical cell of $D(n-1)$. By our inductive hypothesis, the subgraph H of $\tau_{i,n}^{-1}(G)$ induced on $[n] \setminus \{2\}$ satisfies $N_H(1) = [n] \setminus \{1,2\}$ and the subgraph of H induced on $[n] \setminus \{1,2\}$ is a path with i at one end. It follows that $\tau_{i,n}^{-1}(G)$ has the structure described in the second claim. $\square$

As there are $(n-2)!$ M_n-critical cells of $D(n)$, each of which corresponds to a graph with $2n-3$ edges, we have achieved the goal stated at the beginning of this section and obtained a new proof of Theorem 1.2.

Note that if T is M_n-critical with $T_{[n]\setminus\{1\}}$ being the path $2, v_3, \ldots, v_n$ then $T \in \mathcal{T}(P)$, where P is the n-gon whose edges are those in the given path along with $\{1,2\}$ and $\{1, v_n\}$. The internal edges of T are $\{1, v_i\}$ for $2 < i < n$.

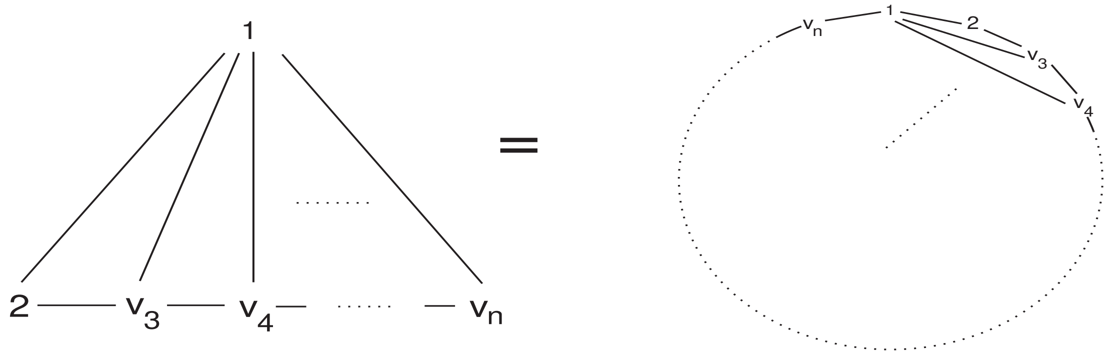


Figure 3: An M_n-critical graph. The dashed part of the ellipse represents the unshown exterior edges in the n-gon with vertices $1, 2, v_3, \ldots, v_n$.

4 The Morse Complex for M_n

In this section we determine, up to signs which depend on a chosen orientation of $\Delta(n)$, the elements $\Phi^{N(M_n)}(G)$ for all M_n-critical cells G of $D(n)$. By Theorems 2.5 and 2.6 and the fact that all M_n-critical cells have dimension $2n-4$, the corresponding elements of the cellular chain complex for $\Delta^2(n)$ form a basis for $\widetilde{H}_{2n-4}(\Delta^2(n))$. The results below will give a proof of Theorem 1.3 and a formula for the sign functions f.

Lemma 4.1 *Let P be an n-gon and let $T \in \mathcal{T}(P)$. Let $e \in E(T)$. Then $T - e$ is 2-connected if and only if e is an internal edge of T.*

Proof: If e is internal then P is a subgraph of $T - e$, and since P is 2-connected so is $T-e$. Now assume e is external. If $n = 3$ then $T-e$ is not 2-connected by observation, so we assume $n \geq 4$. We may assume that $e = \{v_1, v_2\}$. Since e is contained in some triangle of T, we may assume that $\{v_1, v_i\} \in E(G)$ with $3 \leq i < n$. Since T is a triangulation, there is no internal edge of T with endpoints in both $\{v_2, v_3, \ldots, v_{i-1}\}$ and $\{v_1, v_i, v_{i+1}, \ldots, v_n\}$. Thus the only edges with endpoints in both of these sets are e and $\{v_{i-1}, v_i\}$. It follows that v_i is a cutpoint of $T - e$. □

Corollary 4.2 *Let T be a triangulation of an n-gon P and let $E_i(T)$ be the set of internal edges of T. Let ∂ be the boundary map on $\Delta(n)$ with respect to some fixed orientation. Then in the quotient complex $\mathcal{C}(\Delta(n)/\Delta^2_n)$ we have*

$$\partial T = \sum_{e \in E_i(T)} \langle \partial T, T - e \rangle (T - e).$$

Proof: This follows from Lemma 4.1 and the definition of ∂. □

Lemma 4.3 *Let P be an n-gon and let $T \in \mathcal{T}(P)$. Let e be an internal edge of T. Then there is a unique $U \in \mathcal{T}(P) \setminus \{T\}$ such that $T - e$ is a subgraph of U. Also, $T - e = U - f$ for some internal edge f of U.*

Proof: Let $e = \{v_i, v_k\}$ with $i < k - 1$. Since e is an internal edge, there are exactly two triangles of T which contain e. In other words, $N_T(v_i) \cap N_T(v_k) = \{v_j, v_l\}$ for some $j, l \in [n] \setminus \{i, k\}$. Since T is a triangulation we may assume that $i < j < k < l$. The edges of $T - e$ divide the interior of P into $n - 4$ triangles and one 4-gon, whose vertices are v_i, v_j, v_k, v_l. The only elements of $\mathcal{T}(P)$ which contain $T - e$ as a subgraph are T and $U := T - e + \{v_j, v_l\}$. Since $j < k < l$, $f := \{v_j, v_l\}$ is an internal edge of U. □

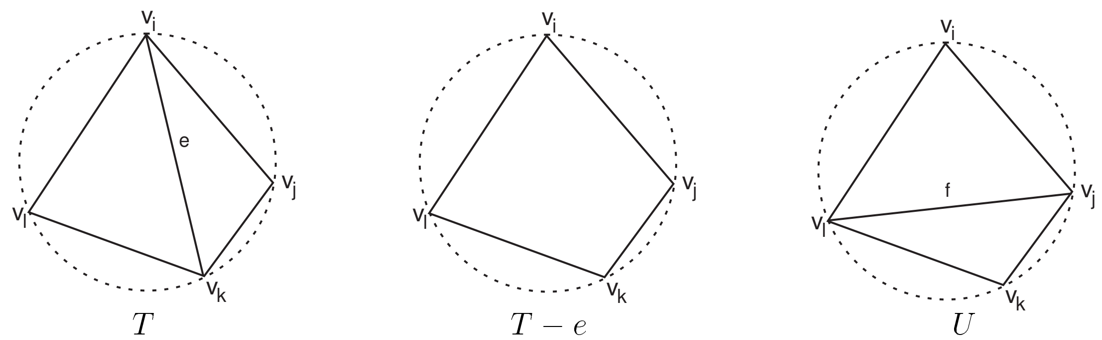


Figure 4: The graphs T, $T - e$ and U of Lemma 4.3.

For the remainder of this section, we assume that P is an n-gon with $\{v_1 = 1, v_2 = 2\} \in E(P)$.

If $P \subseteq G \subseteq T$ with T a triangulation of P, define

$$r(G) := max\,\{j : v_i \in N_G(1) \text{ for } 2 \leq i \leq j\}\,,$$

$$s(G) := max\left\{j : v_j \in N_G(v_{r(G)})\right\},$$

and if $r(G) < n$ define

$$k(G) := min\,\{j > r(G) : v_j \in N_G(1)\}\,.$$

Note that if $T \in \mathcal{T}(P)$ and $r(T) < n$ then $k(T) = s(T)$. Indeed, there is a unique triangle in T containing the edge $\left\{1, v_{r(T)}\right\}$ other than the one whose vertices are $1, v_{r(T)}, v_{r(T)-1}$ (which exists if and only if $r(T) > 2$). In other words, there is a unique $j > r(T)$ such that $v_j \in N_T(1) \cap N_T(v_{r(T)})$. By the definitions, $s(T) \geq j \geq k(T)$. Since T is a triangulation and $\left\{v_{r(T)}, v_{s(T)}\right\} \in E(T)$, we also have $k(T) \geq s(T)$. Thus $s(T) = j = k(T)$.

Lemma 4.4 *Let $T \in \mathcal{T}(P)$. Set $r = r(T)$ and $s = s(T)$. Then*

1. *T is M_n-critical if and only if $r = n$.*
2. *If $r < n$ then $M_n(T) = T - \{v_r, v_s\}$.*

Proof: The first claim follows immediately from Theorem 3.5(2). Now assume $r < n$. We first prove the second claim under the assumption that the claim holds whenever $r = 2$. This proof is by induction on n, with the case $n = 3$ being trivial.

Now assume $n > 3$. Since we are assuming that our claim holds when $r = 2$, we assume that $r > 2$. Then $N_T(2) = \{1, v_3\}$ and $T \in C_n^{(2)}(v_3)$. Thus

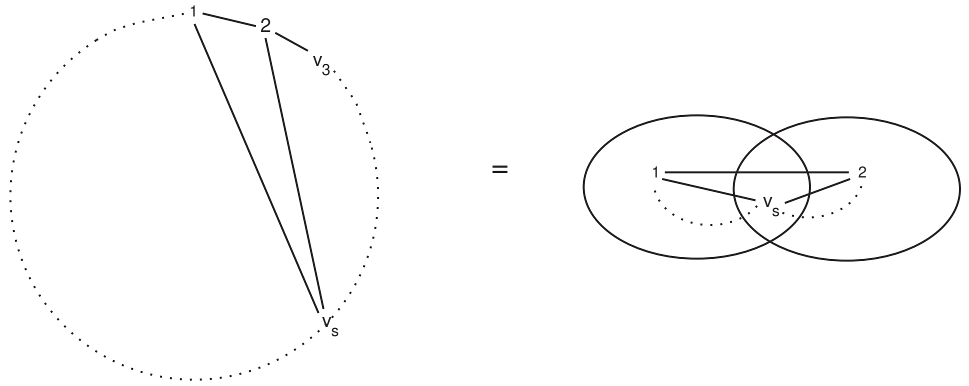
$$M_n(T) = \tau_{v_3,n}^{-1}(M_{n-1}(\tau_{v_3,n}(T))).$$

Let $U = \tau_{v_3,n}(T)$. Then $U \in \mathcal{T}(Q)$, where Q is the $(n-1)$-gon with edges $\{1, 2\}$, $\{2, \phi_{v_3,n}(v_4)\}$, …, $\{\phi_{v_3,n}(v_n), 1\}$. We have $r(U) = r - 1$ and $s(U) = s - 1$. Let $x = \phi_{v_3,n}(v_r)$ and let $y = \phi_{v_3,n}(v_s)$. By inductive hypothesis, $M_{n-1}(U) = U - \{x, y\}$. By definition, $\tau_{v_3,n}^{-1}(U - \{x, y\}) = T - \{v_r, v_s\}$. This completes the inductive step.

We finish by proving the second claim under the assumption that $r = 2$. As $\{1, 2\}$ is contained in some triangle of T we have $s > 3$, so $\{2, v_s\}$ is an internal edge. As noted above, $s = k(T)$ and $v_s \in N_T(1) \cap N_T(2)$. Since T is a triangulation of P, there is no internal edge of T with one endpoint in $\{2, v_3, \ldots, v_{s-1}\}$ and one endpoint in $\{1, v_{s+1}, v_{s+2}, \ldots, v_n\}$. The only external edge with one endpoint in each of these sets is $\{1, 2\}$. Therefore, v_s is a cutpoint of $T - \{1, 2\}$. The subgraph of T induced on $\{1\} \cup \{v_j : j \geq s\}$ contains the cycle $1, v_s, v_{s+1}, \ldots, v_n, 1$ and is thus 2-connected. Therefore, $T \in C_n^{(1)}$ and $b(T) = v_s$. By the definition of M_n,

$$M_n(T) = M_n^{(2)}(T) = T - \{2, v_s\}$$

as claimed. □



$T \in \mathcal{T}(P)$ $\qquad\qquad$ $T \in C_n^{(1)}$, $v_s = b(T)$, $M_n(T) = T - \{2, v_s\}$

Figure 5: The case $r = 2$ in Lemma 4.4(2).

Lemma 4.5 *Let $G = T - e$ where $T \in \mathcal{T}(P)$ and e is an internal edge of T. Set $r = r(G)$, $s = s(G)$ and $k = k(G)$. Then*

1. *If $\{1, v_s\} \in E(G)$ then $M_n(G) = G - \{v_r, v_s\}$.*
2. *If $\{1, v_s\} \notin E(G)$ then e is one of $\{1, v_s\}$ or $\{v_r, v_k\}$ and $M_n(G) = G + \{v_r, v_k\} \in \mathcal{T}(P)$.*

Proof: If $r > 2$ then $G \in C_n^{(2)}(v_3)$ and, after arguing as we did in the proof of Lemma 4.4, we may assume that $r = 2$. If $\{1, v_s\} \in E(G)$ then the argument used in the proof of the second claim of Lemma 4.4 when $r = 2$ can be applied to show that $G \in C_n^{(1)}$ and $b(G) = v_s$. The first claim follows.

Now assume that $\{1, v_s\} \notin E(G)$. Since T is a triangulation of P and $E(G) \subseteq E(T)$, there is no edge $\{1, v_j\} \in E(G)$ with $2 < j < s$. Since $\{1, v_s\} \notin E(G)$, we have $N_G(1) \cap N_G(2) = \emptyset$. However, there is a triangle in T containing the edge $\{1, 2\}$, so $N_T(1) \cap N_T(2) \neq \emptyset$. Thus either $e = \{1, v_s\}$ or $e = \{2, v_j\}$ with $v_j \in N_G(1)$.

In the first case, we have $k > s$ and there is no internal edge $\{v_i, v_j\} \in E(G)$ with $i \in \{1, k+1, k+2, \ldots, n\}$ and $j \in \{2, 3, \ldots, k-1\}$. Thus v_k is a cutpoint of $G - \{1, 2\}$. The subgraph of G induced on $\{1, v_k, v_{k+1}, \ldots, v_n\}$ contains the cycle $1, v_k, \ldots, v_n, 1$ and is thus 2-connected. Therefore $G \in C_n^{(1)}$ and $b(G) = v_k$. Since $k > s$, $\{2, v_k\} \notin E(G)$ and

$$M_n(G) = M_n^{(2)}(G) = G + \{2, v_k\}.$$

Now $\{1, v_s\}$ is an internal edge of T and is therefore contained in two triangles, one of which has vertices $\{1, 2, v_s\}$. The other must have vertices $\{1, v_s, v_j\}$ with $j > s$, and by the definition of $k = k(G)$ we have $j = k$, so $\{v_s, v_k\} \in E(T)$. Therefore, G divides the interior of P into $n - 4$ triangles and one 4-gon, whose vertices are $\{1, 2, v_s, v_k\}$. It follows that $M_n(G) \in \mathcal{T}(P)$.

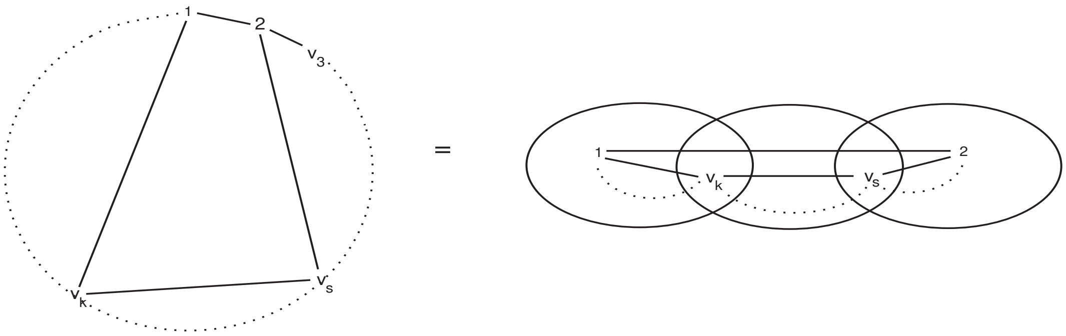
In the second case, since T is a triangulation we have

$$N_G(1) \cap \{3, 4, \ldots, k-1\} = N_G(2) \cap \{k+1, \ldots, n\} = \emptyset,$$

so $j = k$. As before, $G \in C_n^{(1)}$ and $b(G) = v_k$, so

$$M_n(G) = G + \{2, v_k\} = T \in \mathcal{T}(P).$$

In both cases, the second claim of the lemma holds. $\square$



G $\qquad$ $M_n(G) = G + \{2, v_k\}$

Figure 6: The structure of G in Lemma 4.5(2).

Corollary 4.6 *Let $T \in \mathcal{T}(P)$. Let e be an internal edge of T and let $G = T - e$. Set $r = r(T)$ and $s = s(T)$. Then exactly one of the following cases holds.*

1. *We have $e = \{1, v_j\}$ with $3 \le j < r$ and $M_n(G) = G + \{v_{j-1}, v_{j+1}\}$.*
2. *We have $e = \{1, v_r\}$ and $M_n(G) = G + \{v_{r-1}, v_s\}$.*
3. *We have $e = \{1, v_s\}$ with $r < n$ and $M_n(G) = G + \left\{v_r, v_{k(G)}\right\}$.*
4. *We have $e = \{v_r, v_s\}$ and $M_n(G) = T$.*
5. *We have $M_n(G) = G - \{v_r, v_s\}$.*

Proof: If $e = \{1, v_j\}$ with $3 \le j < r$ then $r(G) = j - 1$ and $k(G) = j + 1$. Also, $s(G) = j$ so $\left\{1, v_{s(G)}\right\} \notin E(G)$. By Lemma 4.5(2), $M_n(G) = G + \{v_{j-1}, v_{j+1}\}$.

If $e = \{1, v_r\}$ then $r(G) = r - 1$ and $k(G) = k(T) = s$. Also, $s(G) = r$ so $\{1, v_s(G)\} \notin E(G)$. By Lemma 4.5(2), $M_n(G) = G + \{v_{r-1}, v_s\}$.

If $e = \{1, v_s\}$ with $r < n$ then $r(G) = r$. Thus $k(G)$ is well defined. Since $\left\{1, v_{s(G)}\right\} \notin E(G)$, we have $M_n(G) = G + \left\{v_r, v_{k(G)}\right\}$ by Lemma 4.5(2).

If $e = \{v_r, v_s\}$ then $r(G) = r$ and $k(G) = s$. Also, $r < s(G) < s$, so $\left\{1, v_{s(G)}\right\} \notin E(G)$. By Lemma 4.5(2), $M_n(G) = G + e = T$.

Finally, assume that e is not one of the edges described in cases (1)-(4). Then $r(G) = r$ and $s(G) = s$. Since $e \neq \{1, v_s\}$, we have $\left\{1, v_{s(G)}\right\} \in E(G)$. By Lemma 4.5(1), $M_n(G) = G - \{v_r, v_s\}$. □

Lemma 4.7 *Let $U \in \mathcal{T}(P)$ with $|N_U(1)| < n-1$. There is a unique $T \in \mathcal{T}(P) \setminus \{U\}$ such that $\langle \partial T, M_n(U)\rangle \neq 0$. This T satisfies $|N_T(1)| = |N_U(1)| + 1$.*

Proof: Let $r = r(U)$, $s = s(U)$ and $e = \{v_r, v_s\}$. By Lemma 4.4, $M_n(U) = U - e$, and e is an internal edge of U. By Lemma 4.3 there is a unique $T \in \mathcal{T}(P) \setminus \{U\}$ such that $U - e$ is a subgraph of T, and $U - e$ is obtained from T by removing an internal edge $f \in E(T)$. The first claim follows. Since e is an internal edge of U, it is contained in exactly two triangles, one of which has vertices $\{1, v_r, v_s\}$. Since U is a triangulation, the other must have vertices $\{v_r, v_q, v_s\}$ with $r < q < s$. It follows that $f = \{1, v_q\}$, giving the second claim. □

Define T_0 to be the unique M_n-critical cell in $\mathcal{T}(P)$. For graphs G, H on the same vertex set V, denote by $G \cap H$ the graph on V whose edge set is $E(G) \cap E(H)$. For $2 \leq j \leq n-1$ define

$$\mathcal{T}_j(P) := \{T \in \mathcal{T}(P) : |N_T(1)| = j\}.$$

For $T \in \mathcal{T}(P)$, define

$$\mathcal{M}(T) := \{U \in \mathcal{T}(P) : T \cap U = M_n(U)\},$$

and

$$\mathcal{M}'(T) := \mathcal{M}(T) \setminus \{T\}.$$

Lemma 4.8 *1. We have $\mathcal{T}_{n-1}(P) = \{T_0\}$.*

2. For $2 \leq j < n-1$,

$$\mathcal{T}_j(P) = \bigcup_{T \in \mathcal{T}_{j+1}(P)} \mathcal{M}'(T),$$

and this union is disjoint.

3. We have

$$\Phi(T_0) = T_0 - \sum_{U \in \mathcal{T}_{n-2}(P)} \langle \partial T_0, T_0 \cap U\rangle \langle \partial U, T_0 \cap U\rangle U.$$

4. For $T \in \mathcal{T}(P) \setminus \{T_0\}$,

$$\Phi(T) = -\sum_{U \in \mathcal{M}'(T)} \langle \partial T, T \cap U\rangle \langle \partial U, T \cap U\rangle U.$$

Proof: The first claim is clear by observation and the second follows from Corollary 4.6 and Lemma 4.7.

By Lemma 4.4, if $T \in \mathcal{T}(P)$ then $\mathcal{V}(T) = 0$, so

$$\Phi(T) = T + \mathcal{V}(\partial T).$$

Let $X_0 = \{G : \langle \partial T_0, G \rangle \neq 0\}$. By Corollary 4.2, $X_0 = \{T_0 - \{1, v_j\} : 2 < j < n\}$. The third claim now follows from Corollary 4.6(1) and the definition of $\mathcal{V}$. The fourth claim follows from the definition of $\mathcal{V}$. □

Let $\Gamma(P)$ be the digraph whose vertices are (indexed by) the elements of $\mathcal{T}(P)$ with an arc (T, U) in $A(\Gamma(P))$ if and only if $U \in \mathcal{M}'(T)$. By induction on j and Lemma 4.7, if $T \in \mathcal{T}_{n-1-j}(P)$ with $1 \leq j \leq n-3$ then there is a unique directed path π from T_0 to T in $\Gamma(P)$, and this path has length j. For each arc $a = (U, W)$ in π, set

$$\alpha(a) := \langle \partial U, U \cap W \rangle \langle \partial W, U \cap W \rangle,$$

and define

$$f(T) := (-1)^j \prod_{a \in \pi} \alpha(a).$$

Also, set $f(T_0) = 1$. Note that if $(T, U) \in A(\Gamma(P))$ then

$$\langle \Phi(T), U \rangle = -\langle \partial T, T \cap U \rangle \langle \partial U, T \cap U \rangle.$$

Theorem 4.9 *1. For $1 \leq j \leq n-3$, we have*

$$\Phi^j(T_0) = \sum_{i=0}^{j} \sum_{T \in \mathcal{T}_{n-1-i}(P)} f(T)T.$$

2. Let $N = N(M_n)$. Then

$$\Phi^N(T_0) = \Phi^{n-3}(T_0) = \sum_{T \in \mathcal{T}(P)} f(T)T.$$

Proof: We prove the first claim by induction on j. When $j = 1$ the claim is just a restatement of Lemma 4.8(3), so assume $j > 1$. Using the inductive hypothesis and Lemma 4.8(2),(4) we get

$$\begin{aligned}
\Phi^j(T_0) &= \Phi(\Phi^{j-1}(T_0)) \\
&= \Phi\left(\sum_{i=0}^{j-2} \sum_{T \in \mathcal{T}_{n-1-i}(P)} f(T)T\right) + \sum_{T \in \mathcal{T}_{n-j}(P)} f(T)\Phi(T) \\
&= \Phi(\Phi^{j-2}(T_0)) + \sum_{T \in \mathcal{T}_{n-j}(P)} f(T) \sum_{U \in \mathcal{M}'(T)} -\alpha((T, U))U \\
&= \Phi^{j-1}(T_0) + \sum_{U \in \mathcal{T}_{n-j-1}(P)} f(U)U \\
&= \sum_{i=0}^{j} \sum_{T \in \mathcal{T}_{n-1-i}(P)} f(T)T
\end{aligned}$$

as claimed.

The second equality in the second claim follows from the first claim. To prove the first equality it is sufficient to show that $\Phi^{n-2}(T_0) = \Phi^{n-3}(T_0)$. Using the first claim, we get

$$\begin{aligned} \Phi^{n-2}(T_0) &= \Phi(\Phi^{n-3}(T_0)) \\ &= \Phi\left(\sum_{j=3}^{n-1} \sum_{T \in \mathcal{T}_j(P)} f(T)T\right) + \sum_{T \in \mathcal{T}_2(P)} f(T)\Phi(T) \\ &= \Phi(\Phi^{n-4}(T_0)) + \sum_{T \in \mathcal{T}_2(P)} f(T)\Phi(T). \end{aligned}$$

Let $T \in \mathcal{T}_2(P)$, so $N_T(1) = \{2, v_n\}$. Let e be an interior edge of T and let $G = T - e$. Then by Corollary 4.6 either $M_n(G)$ is a subgraph of G or $e = \{v_r(T), v_s(T)\} = \{2, v_n\}$ and $M_n(G) = T$. Thus $\mathcal{M}'(T) = \emptyset$ and $\Phi(T) = 0$ by Lemma 4.8(4). The second claim now follows. $\square$

Theorem 1.3 now follows from Theorems 2.5, 2.6 and 4.9. In closing, I note that once the Morse matching M_n has been found, there is an approach to proving Theorem 1.3 other than the one given in this section. This alternative approach was suggested by Wachs. One can show that if $G \in V(n)$ satisfies $P \subseteq G \subseteq T$ for some n-gon P with $\{1, 2\} \in E(P)$ and some triangulation T of P, then $P \subseteq M_n(G) \subseteq T$. Let $\Sigma(n)$ be the subcomplex of $\Delta^2(n)$ consisting of the point p_0 (corresponding to the subcomplex Δ^2_n of $\Delta(n)$) and all G satisfying $P \subseteq G \subseteq T$ for some P, T as above. Then basic arguments from Forman's theory show that $\Delta^2(n)$ collapses onto $\Sigma(n)$, so $\Sigma(n)$ has the same homology as $\Delta^2(n)$. The classes c_P of Theorem 1.3 cannot be boundaries in $\Sigma(n)$, since their support consists of maximal cells. Since their supports are disjoint, the c_P (for $P \in \mathcal{P}_n$) are linearly independent, and it remains to show that they are cycles. To do this, fix an n-gon P. The triangulations of P are in natural bijective correspondence with the maximal faces of the associahedron of degree n (see [4]). Using Lemma 4.1, one sees that with respect to some orientation of $\Delta(n)$, the sum of these triangulations corresponds to the fundamental cycle of the associahedron. I have chosen to present the proof of this section in order to illustrate the power of Forman's theory, which may prove useful in other settings.

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