Abstracts of papers by John Shareshian

Published work

1. Complexes of not i-connected graphs (with E. Babson, A. Björner, S. Linusson, V. Welker), *Topology* **38** (1999), no. 2, 271–299.

   Complexes of (not) connected graphs, hypergraphs and their homology appear in the construction of knot invariants given by V. Vassiliev. In this paper we study the complexes of not i-connected k-hypergraphs on n vertices. We show that the complex of not 2-connected graphs has the homotopy type of a wedge of $(n-2)!$ spheres of dimension $2n-5$. This answers a question raised by Vassiliev in connection with knot invariants. For this case the S_n-action on homology of the complex is also determined. For complexes of not 2-connected k-hypergraphs we provide a formula for the generating function of the Euler characteristic, and we introduce certain lattices of graphs that encode their topology. We also present partial results for some other cases. In particular, we show that the complex of not $(n-2)$-connected graphs is Alexander dual to the complex of partial matchings of the complete graph. For not $(n-3)$-connected graphs we provide a formula for the generating function of the Euler characteristic.

2. Links in the complex of separable graphs, *J. Combin. Theory Ser. A* **88** (1999), no. 1, 54–65.

   We determine the homotopy type of the link of a nonempty face in the complex of not 2-connected graphs on n vertices. Corollaries of our main result are that the complex of not 2-connected 3-uniform hypergraphs on n vertices has the homtopy type of a wedge of spheres in dimension $n-4$ (this has consequences in V. Vassiliev's theory of invariants of ornaments) and that the lattice of block-closed graphs on n vertices is Cohen-Macaulay.

3. On the probabilistic zeta function for finite groups, *J. Algebra* **210** (1998), no. 2, 703–707.

   Let G be a finite group and define the function

   $$P(G,s) := \sum_{H \leq G} \frac{\mu(H,G)}{[G:H]^s},$$

   where μ is the Möbius function on the subgroup lattice of G. The function $P(G,s)$ is an inverse for a zeta function of G, as described by A. Mann and N. Boston. Boston conjectured that $P'(G,1) = 0$ if G is nonabelian simple. We prove a generalization of this conjecture, showing that $P'(G,1) = 0$ unless $G/O_p(G)$ is cyclic for some prime p.

4. On the Möbius number of the subgroup lattice of the symmetric group, *J. Combin. Theory Ser. A* **78** (1997), no. 2, 236–267.

   In this paper, a formula is given for the Möbius number $\mu(1,S_n)$ of the subgroup lattice of the symmetric group S_n. This formula involves the Möbius numbers of certain transitive subgroups of S_n. When n has at most two (not necessarily distinct) prime factors or n is a power of two, this formula is refined so that it involves only the Möbius numbers of certain primitive subgroups of S_n. Using the O'Nan-Scott Theorem, the classification of finite simple groups, and the refined formula, the exact value of $\mu(1,S_n)$ is determined when n is prime, twice a prime, or a power of two. For certain primes p, $|\mu(1,S_{2p})| \neq \frac{(2p)!}{2}$. This result gives a negative answer to a question raised by R. Stanley.

Papers to appear

5. Discrete Morse theory for complexes of 2-connected graphs, To appear in *Topology.*

   Using the discrete Morse theory of R. Forman, we find a basis for the unique nonzero homology group of the complex of 2-connected graphs on n vertices. This answers a question of V. Vassiliev which arises in his study of knot invariants.

6. On the shellability of the order complex of the subgroup lattice of a finite group, To appear in *Trans. Amer. Math. Soc.*

   We show that the order complex of the subgroup lattice of a finite group G is shellable if and only if G is solvable. A by-product of the proof that nonsolvable groups do not have shellable subgroup lattices is the determination of the homotopy types of the order complexes of the subgroup lattices of many minimal simple groups.

Submitted papers

7. Complexes of t-colorable graphs (with S. Linusson).

   We study the simplicial complex of t-colorable graphs on n vertices. We prove this complex is homotopy equivalent to a wedge of spheres all of dimension $n(t-1) - \binom{t}{2} - 1$ when $t = 2$ and when $t \geq n - 3$. We show that such a homotopy equivalence does not hold for general t and n.

8. Enumerating representations in finite wreath products II: Explicit formulae (with T. Müller).

   We present several contributions to the enumerative theory of wreath product representations developed in a previous paper [M] of the first author. One of the key ingredients in the description of the generating functions arising in this context (the exterior function Φ_Γ) is calculated. Building on Theorem 1 of [M] and this result we derive explicit formulae for the generating functions $\sum |\mathrm{Hom}(\Gamma, \mathcal{R}_n)| \, z^n/n!$, where Γ is dihedral or a finite abelian group, and the representation sequence $\{\mathcal{R}_n\}$ is any of $\{H \wr S_n\}$ or $\{H \wr A_n\}$ with a finite group H, or the sequence $\{W_n\}$ of Weyl groups of type D_n. Moreover, we verify a conjecture concerning the asymptotic behaviour of the sequence $\{|\mathrm{Hom}(G, W_n)|\}$ for finite groups G in the case when G is dihedral or abelian.

Preprints

9. Nontrivial homology in the subgroup lattices of symmetric and alternating groups.

   For a finite group G, let $\Delta(G)$ be the simplicial complex whose k-dimensional faces correspond to chains

   $$H_0 < H_1 < \ldots < H_k$$

   of nontrivial proper subgroups of G. C. Kratzer and J. Thèvenaz have determined the homotopy type of $\Delta(G)$ when G is solvable, but little is known about the topology of $\Delta(G)$ when G is not solvable. Here we show that for any $n > 2$ both $\widetilde{H}_{n-3}(\Delta(S_n))$ and $\widetilde{H}_{n-4}(\Delta(A_n))$ are nontrivial. Using the EL-shellability theory of A. Björner and M. Wachs, we describe some nontrivial elements of these homology groups. In addition, we determine the homotopy type of $\Delta(S_p)$ for all primes p and the homotopy type of $\Delta(A_p)$ for most primes p.

10. The Lefschetz character for the action of a finite general linear group on the lattice of partial direct sums

    The lattice $\mathcal{PD}(n,q)$ is the poset whose elements are all collections $\{W_1,\ldots,W_k\}$ of subspaces of a fixed n-dimensional vector space V over $\mathbf{F}_q$ such that the sum $\sum_{i=1}^{k} W_i$ is direct, ordered by refinement. That is, $\{W_1,\ldots,W_k\} \leq \{X_1,\ldots,X_l\}$ if each W_i is contained in some X_j. This lattice admits a natural action of $GL_n(q)$. It is shown herein that, up to a sign, the Lefschetz character for this action is induced from a linear character τ of the normalizer of a Coxeter torus C of $GL_n(q)$, where $ker(\tau) = C$ and τ is faithful on a complement to C. H. Crapo's closure theorem and induction on n are used to reduce this claim to an identity in the character ring of $GL_n(q)$ involving the Steinberg character and linear characters induced from normalizers of maximal tori, and a theorem of B. Srinivasan is used to verify that this identity holds by examining an identity in the character ring of S_n. The result bears a remarkable resemblance to the well-known result of R. Stanley on the Lefschetz character for the action of S_n on the lattice of partitions of an n-set.

11. Quillen complexes and hypergraph matching complexes.

Let G be a finite group and let p be a prime dividing $|G|$. The Quillen complex $\Delta(\mathcal{A}_p(G))$ is the simplicial complex whose k-dimensional faces are all chains

$$E_0 < E_1 < \ldots < E_k$$

of nontrivial elementary abelian p-subgroups of G. Here we examine the case where G is a symmetric group. In the thesis of R. Ksontini ([Ks]), information about the homology and fundamental group of $\Delta(\mathcal{A}_p(S_n))$ is obtained by examining two related complexes. The first is the complex $\mathcal{CT}_p(n)$ whose k-dimensional faces are collections

$$\{\langle\sigma_1\rangle, \ldots, \langle\sigma_{k+1}\rangle\}$$

of cyclic subgroups of S_n which are generated by p-cycles σ_i, such that if $i \neq j$ then σ_i and σ_j have disjoint support. The second is the complex $\mathcal{CD}_p(n)$ whose k-dimensional faces are collections

$$\{X_1, \ldots, X_{k+1}\}$$

of subsets of $\{1, \ldots, n\}$ such that $|X_i| = p$ for all i and $X_i \cap X_j = \emptyset$ whenever $i \neq j$.

Here we extend various results from [Ks]. One of our main tools is a theorem of A. Björner. M. Wachs and V. Welker, which we use to show that $\mathcal{CT}_p(n)$ is homotopy equivalent to a wedge of complexes, each of which can be built from spheres and the complexes $\mathcal{CD}_p(m)$ with $m \leq n$ and $m \equiv n \bmod p$, using the wedge and join operations. In particular, the homology of $\mathcal{CD}_p(n)$ embeds in that of $\mathcal{CT}_p(n)$. This allows us to determine the fundamental group of $\Delta(\mathcal{A}_p(S_n))$ when $n = 3p$, settling one of the two cases left open in [Ks] (the other is $n = 3p+1$). Next, using standard results from homotopy theory, we show that when $p^2 + p \leq n < 2p^2 - p$, the complex $\Delta(\mathcal{A}_p(n))$ is homotopy equivalent to a wedge of complexes, one of which is is $\mathcal{CT}_p(n)$. Note that similar results are proven in [Ks] when $p^2 \leq n < p^2 + p$, and it is straightforward to show that $\Delta(\mathcal{A}_p(S_n))$ is homotopy equivalent to $\mathcal{CT}_p(n)$ when $n < p^2$. Computer calculations indicate that $\widetilde{H}_2(\mathcal{CD}_3(13))$ contains an element of order two. It follows that $\widetilde{H}_2(\Delta(\mathcal{A}_3(S_{13})))$ has an element of order two. This seems to be the first example of torsion in the Quillen complex of a finite group.

Papers in preparation

12. Symmetric and alternating groups acting on k-sets as monodromy groups (with R. Guralnick).

    We investigate analytic maps $f : X \to \mathbf{P}^1$ such that the associated monodromy group is a symmetric group S_n or an alternating group A_n acting on subsets of fixed size k (with $2 \leq k \leq n-2$) from $\{1, \ldots, n\}$. Using the Riemann-Hurwitz formula and various combinatorial lemmas, we show that if f has at least five branch points then, with a short list of known exceptions, the genus $g(X)$ must be large (in an appropriate sense) with respect to both the number of branch points and the number of sheets of the associated branched covering of $\mathbf{P}^1$. Among the consequences of our work are the following results.

    - For $G \leq S_n$ and integers g, r let $\mathcal{M}(G, g, r)$ be the moduli space of all (equivalence classes of) genus g Riemann surfaces X which admit an analytic map $f : X \to \mathbf{P}^1$ with r branch points and monodromy group G (acting on n sheets). If $g > 4$, $r \geq 5$ and $\mathcal{M}(G, g, r)$ is dense in the moduli space $\mathcal{M}_g$ of all Riemann surfaces of genus g, then $G = A_n$ or $G = S_n$
    - For $G \leq S_n$ and $k < n$ let $\mathcal{T}_k(G)$ be the set of all equivalence classes of trinomials $f(x) = ax^n + bx^k + c$ with $a, b, c \in \mathbf{Q}$ such that the Galois group of f is contained in G, where equivalence is given by $f(x) \sim f(\alpha x)$ for any constant α. If $n > 10$ then one of the following conditions holds.
        - G is one of S_n, A_n, or the stabilizer of a point, a 2-set or an ordered pair in the natural action of S_n.
        - $\mathcal{T}_k(G)$ is finite.

13. Top and bottom homology of matching and chessboard complexes (with M. Wachs)

For a positive integer n, the matching complex M_n is the simplicial complex whose vertices are (in correspondence with) the edges of the complete graph K_n, and whose k-dimensional faces are matchings (that is, subgraphs in which no vertex is contained in more that one edge) from K_n with $k+1$ edges. Matching complexes arise naturally in algebraic topology, finite group theory, combinatorics, commutative algebra and algebraic geometry and were originally studied in the papers of S. Bouc [Bo] and A. Björner, L. Lovász, S. Vrećica and R. Živaljević [BLVZ]. In both of these papers, it is shown that if one defines

$$\nu = \nu(n) := \lfloor \frac{n+1}{3} \rfloor$$

then M_n is $(\nu-2)$-connected. In [BLVZ] the authors conjecture that M_n is not $(\nu-1)$-connected. Using methods developed in [Bo], we prove this conjecture. In addition, we show that with finitely many exceptions the first nontrivial homology group $\widetilde{H}_{\nu-1}(M_n)$ is an elementary abelian 3-group. Also, we show that the $(\nu-1)$-skeleton of M_n is shellable, settling another conjecture from [BLVZ].

For positive integers m, n, the chessboard complex $M_{m,n}$ is the simplicial complex whose k-dimensional faces are all sets of $k+1$ squares from an $m \times n$ chessboard such that no row or column of the chessboard contains more than one square from the given set (equivalently, $M_{m,n}$ is the matching complex on the complete bipartite graph $K_{m,n}$). Chessboard complexes arise in both combinatorics and commutative algebra and were also studied in [BLVZ]. Assuming $m \leq n$, define

$$\nu = \nu(m,n) := \min(m, \lfloor \frac{m+n+1}{3} \rfloor).$$

In [BLVZ] it is shown that $M_{m,n}$ is $(\nu-2)$-connected and conjectured that $M_{m,n}$ is not $(\nu-1)$-connected for all m, n. We prove this conjecture and show that in many cases $\widetilde{H}_{\nu-1}(M_{m,n})$ is finite, with torsion at the primes $2, 3$ only. A key ingredient in our proof is the use of the Robinson-Schensted algorithm to determine a basis for the top homology group of $M_{m,n}$.

14. Topology of some coset posets

For a finite group G, let $\mathcal{C}(G)$ bet the set of all cosets of all proper subgroups of G, partially ordered by inclusion. Let $\Delta(\mathcal{C}(G))$ be the order complex of this poset, so k-dimensional simplices in $\Delta(\mathcal{C}(G))$ are chains

$$X_0 \subset X_1 \subset \ldots \subset X_k$$

from $\mathcal{C}(G)$. The study of these complexes was initiated by K. S. Brown in [Br], where it is shown that the reduced Euler characteristic of $\Delta(\mathcal{C}(G))$ is equal to $P(G, -1)$, where $P(G, s)$ is the probabalistic zeta function for G. It is also shown in [Br] that if G is solvable then $\Delta(\mathcal{C}(G))$ has the homotopy type of a wedge of spheres, the dimension of each of these spheres being one less than the number of complemented chief factors in G. Here we determine the homotopy type of $\Delta(\mathcal{C}(G))$ for some simple and almost simple groups G. More precisely, let p be a prime. If $G = L_2(2^p)$ then $\Delta(\mathcal{C}(G))$ has the homotopy type of a wedge of 2-spheres. If in addition every transitive proper subgroup of A_p is solvable (a condition which holds for most primes p) then $\Delta(\mathcal{C}(A_p))$ and $\Delta(\mathcal{C}(S_p))$ both have the homotopy type of a wedge of spheres. The spheres have dimensions 2 and $p-3$ for A_p and dimensions 2 and $p-2$ for S_p. So, it is not the case that $\Delta(\mathcal{C}(G))$ has the homotopy type of a wedge of spheres all of one dimension for every finite group G, and this unidimensional condition does not characterize solvability. The key tool in our proofs is the shellability theory of A. Björner and M. Wachs.

15. A Lefschetz character for the finite projective linear and unitary groups in prime dimension and the Möbius numbers of $PGL_3(q)$, $L_3(q)$, $PGU_3(q)$ and $U_3(q)$

    Let G be a projective linear or unitary group in odd prime dimension over a finite field of odd order. Let $\mathcal{I}$ be the poset of nontrivial subgroups of G which act reducibly on the natural projective space for G. We show that the Lefschetz character for the action of G on $\mathcal{I}$ by conjugation is induced from a linear character τ of the normalizer of a Coxeter torus $C \leq G$, where $ker(\tau) = C$ and τ is faithful on a complement to C. This result is remarkably similar to a natural restatement of the well-known result of Stanley on the Lefschetz character for the action of S_n on the lattice of partitions of an n-set. Using this result, we are able to determine the Möbius number $\mu(1, G)$ of the subgroup lattice of G when G is one of the three dimensional projective groups listed in the title. With few exceptions, $\mu(1, G) = 0$.

16. Möbius numbers and Lefschetz characters of finite orthogonal groups in odd prime dimension

    For an odd prime d and and odd prime power q, let G be one of $SO_d(q)$ or $\Omega_d(q)$. If $q = 3$ assume $G = SO_d(q)$. We determine the Lefschetz character for the action of G by conjugation on the poset of nontrivial subgroups of G which act reducibly on the natural vector space for G. If $G = SO_d(q)$, this character is induced from a faithful character of a cyclic subgroup of G which regularly permutes an orthogonal frame. If $G = \Omega_d(q)$ then this character is the sum of two such induced characters. Using this result, we derive a formula for the Möbius number $\mu(1, G)$ of the subgroup lattice of G whenever the alternating group on d letters does not contain any nonsolvable transitive proper subgroups. This formula involves the Möbius numbers of proper subgroups of G which lie in Aschbacher's class $\mathcal{C}_9$ of primitive, absolutely irreducible subgroups which are not defined over a proper subfield of $\mathbf{F}_q$.