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ON THE SHELLABILITY OF THE ORDER COMPLEX OF THE SUBGROUP LATTICE OF A FINITE GROUP

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Abstract. In this paper it will be shown that the order complex of the subgroup lattice of a finite group G is nonpure shellable if and only if G is solvable. A by-product of the proof that nonsolvable groups do not have shellable subgroup lattices is the determination of the homotopy types of the order complexes of the subgroup lattices of many minimal simple groups.

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1. Introduction

In this paper it will be shown that the order complex of the subgroup lattice of a finite group G is (nonpure) shellable if and only if G is solvable. The proof of nonshellability in the nonsolvable case involves the determination of the homotopy type of the order complexes of the subgroup lattices of many minimal simple groups.

We begin with some history and basic definitions. It is assumed that the reader is familiar with some of the rudiments of algebraic topology and finite group theory. No distinction will be made between an abstract simplicial complex Δ and an arbitrary geometric realization of Δ. Maximal faces of a simplicial complex Δ will be called facets of Δ.

Definition 1.1. A simplicial complex Δ is **shellable** if the facets of Δ can be ordered $\sigma_1, \dots, \sigma_n$ so that for all $1 \leq i < k \leq n$ there exists some $1 \leq j < k$ and $x \in \sigma_k$ such that $\sigma_i \cap \sigma_k \subseteq \sigma_j \cap \sigma_k = \sigma_k \setminus \{x\}$. The list $\sigma_1, \dots, \sigma_n$ is called a shelling of Δ.

Note that this definition does not require that Δ be pure. Pure shellable complexes are discussed in [1], and nonpure shellable complexes are introduced in [2].

Definition 1.2. Let P be a partially ordered set. The order complex $\Delta(P)$ is the simplicial complex whose k-simplices are chains of length k from P.

Note that a chain of length k from a poset P contains $k+1$ elements. We will call P shellable if and only if $\Delta(P)$ is shellable.

Definition 1.3. Let G be a finite group.

1. $\mathcal{L}(G)$ is the lattice of subgroups of G.
2. $\overline{\mathcal{L}(G)} := \mathcal{L}(G) \setminus \{1, G\}$.
3. $l(G)$ is the length of the longest chain in $\overline{\mathcal{L}(G)}$.

It follows immediately from Definition 1.1 that $\mathcal{L}(G)$ is shellable if and only if $\overline{\mathcal{L}(G)}$ is shellable. We will call G shellable if and only if $\overline{\mathcal{L}(G)}$ is shellable. As stated above, we will prove the following theorem.

Theorem 1.4. *A finite group G is shellable if and only if G is solvable.*

The relation between subgroup lattices and pure shellability is already well-understood. Note that for any poset P, the order complex $\Delta(P)$ is pure if and only if P is graded, that is, every maximal chain in P has the same length. In [10], Iwasawa showed that for a finite group G, $\mathcal{L}(G)$ is graded if and only if G is supersolvable. In [13], Stanley introduced supersolvable lattices, which are lattices which share some important combinatorial properties with subgroup lattices of finite supersolvable groups. In [1], Björner showed that supersolvable lattices are shellable, thereby showing that the three conditions G is supersolvable, $\Delta(\overline{\mathcal{L}(G)})$ is pure, and $\Delta(\overline{\mathcal{L}(G)})$ is shellable and pure are equivalent.

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2. Homotopy of order complexes and properties of shellable complexes

It this section we will review some known results about order complexes and shellable complexes which will be used throughout the paper. The reader who is aware of the main results in these areas can skip this section and refer back to it when necessary. We begin with some homotopy results of Quillen.

Definition 2.1. Let P, Q be posets.

1. A morphism $f : P \to Q$ is a function such that $x \leq_P y$ implies $f(x) \leq_Q f(y)$.
2. If $f : P \to Q$ is a morphism, then for $y \in Q$ define
$$f_y := \{x \in P : f(x) \leq y\}$$
and
$$f^y := \{x \in P : f(x) \geq y\}.$$

Proposition 2.2 ([11], 1.5)**.** *If a poset P contains a unique minimal element or a unique maximal element then $\Delta(P)$ is contractible.*

Proposition 2.3 ([11], Proposition 1.6)**.** *Let $f : P \to Q$ be a morphism of posets. If $\Delta(f_y)$ is contractible for all $y \in Q$ or $\Delta(f^y)$ is contractible for all $y \in Q$ then f induces a homotopy equivalence between $\Delta(P)$ and $\Delta(Q)$.*

Corollary 2.4. *Let P be a finite poset. Set*
$$S = \{x \in P : x \text{ is covered by a unique element in } P\}$$
and
$$T = \{x \in P : x \text{ covers a unique element in } P\}.$$
Then for any $S' \subseteq S$ and $T' \subseteq T$, $\Delta(P)$ is homotopy equivalent to both $\Delta(P \setminus S')$ and $\Delta(P \setminus T')$.

Proof. Let $i : P \setminus S' \hookrightarrow P$ be the identity embedding. Fix $y \in P$. If $y \notin S'$ then $i^y = \{x \in P \setminus S' : x \geq y\}$. If $y \in S'$, let z_0 be the unique element of P covering y. For $i \geq 0$, if $z_i \in S'$ let z_{i+1} be the unique element of P covering z_i. Since P is finite, there is a smallest k such that $z_k \notin S'$, and $i^y = \{x \in P \setminus S' : x \geq z_k\}$. In any case, $\Delta(i^y)$ is contractible by Proposition 2.2, and i induces a homotopy equivalence of order complexes by Proposition 2.3. The proof for $P \setminus T'$ is similar. ☐

Corollary 2.5. *Let L be a finite lattice and let M be the sublattice of L consisting of the minimum element $\hat{0}$, the maximum element $\hat{1}$, and all $x \in L$ such that $x = \bigwedge_{c \in C} c$ for some set C of coatoms of L. For $X \in \{L, M\}$ let $\overline{X} = X \setminus \{\hat{0}, \hat{1}\}$. Then $\Delta(\overline{L})$ and $\Delta(\overline{M})$ are homotopy equivalent.*

Proof. Let $i : \overline{M} \to \overline{L}$ be the identity embedding, and let $\mathcal{C}$ be the set of coatoms of L. For each $x \in \overline{L}$, let $\mathcal{C}(x) = \{c \in \mathcal{C} : x \leq c\}$. Then $\bigwedge_{c \in \mathcal{C}(x)} c$ is the unique minimum element of i^x. The corollary now follows from Propositions 2.2 and 2.3. ☐

Next we state a well-known collapsibility result (see [8]).

Lemma 2.6. *Let Δ be a simplicial complex and let τ be a face of Δ such that τ is properly contained in a unique facet of Δ. Let Δ_τ be the complex obtained by removing from Δ all faces containing τ. Then Δ and Δ_τ are homotopy equivalent.*

The following results on shellable complexes appear in [2].

Definition 2.7. Let Δ be a finite simplicial complex.

1. For a simplex $\sigma \in \Delta$, $d(\sigma) := max\left\{|\tau| : \sigma \subseteq \tau \in \Delta\right\}$.
2. For $0 \leq j \leq i$, $f_{i,j}$ is the number of simplices $\sigma \in \Delta$ such that $|\sigma| = j$ and $d(\sigma) = i$.
3. For $0 \leq j \leq i$, $h_{i,j} := \sum_{k=0}^{j}(-1)^{j-k}\binom{i-k}{j-k} f_{i,k}$.

Definition 2.8. Let Δ be a finite simplicial complex and let $\sigma_1, \dots, \sigma_n$ be a shelling of Δ.

1. $R(\sigma_k) := \{x \in \sigma_k : \sigma_k \setminus \{x\} \subseteq \sigma_i \text{ for some } i < k\}$.
2. $\Gamma_j := \{\sigma_k : |\sigma_k| = j \text{ and } R(\sigma_k) = \sigma_k\}$.

Theorem 2.9 ([2], Theorems 3.4 and 4.1)**.** *Let Δ be a finite simplicial complex and let $\sigma_1, \dots, \sigma_n$ be a shelling of Δ.*

1. *Then $h_{i,j}$ is the number of σ_k such that $|\sigma_k| = i$ and $|R(\sigma_k)| = j$. In particular, $h_{j,j} = |\Gamma_j|$ and $h_{i,j} \geq 0$ for all i, j.*
2. *Also, Δ has the homotopy type of a wedge of spheres, consisting of $h_{j,j}$ spheres of dimension $j-1$ for each j. In particular, $dim(\tilde{H}_{j-1}(\Delta)) = h_{j,j}$ is at most the number of facets of Δ having dimension $j-1$.*

Definition 2.10. Let Δ be an abstract simplicial complex of finite dimension d. For each $0 \leq r \leq s \leq d$, define the complex

$$\Delta^{(r,s)} := \{\tau \in \Delta : dim(\tau) \leq s \text{ and } \tau \subseteq \sigma \text{ for some } \sigma \in \Delta \text{ with } dim(\sigma) \geq r\}.$$

Theorem 2.11 ([2], Theorem 2.9)**.** *If Δ is a shellable complex of finite dimension d, then $\Delta^{(r,s)}$ is shellable for all $0 \leq r \leq s \leq d$.*

Definition 2.12. Let P be a finite poset with a unique minimum element $\hat{0}$ and a unique maximum element $\hat{1}$. We say that P admits a recursive coatom ordering if $P = \{\hat{0}, \hat{1}\}$ or if there is an ordering $m_1, \dots, m_r$ of the coatoms of P which satisfies the following two properties.

(R) For all $j \in [r]$, the poset $P_j = [\hat{0}, m_j]$ admits a recursive coatom ordering in which the coatoms of P_j which are contained in a poset $P_i = [\hat{0}, m_i]$ for some $i < j$ come before all other coatoms.

(S) For all $1 \leq i < k \leq r$ and $x \in P$, if $x \leq m_i$ and $x \leq m_k$ then there exist some $j < k$ and some coatom w of P_k such that $x \leq w \leq m_j$.

Since the posets we will work with are lattices, we can replace condition (S) in Definition 2.12 by the equivalent condition described below. The equivalence of the given conditions follows immediately from the definition of a lattice.

Proposition 2.13. *Let L be a finite lattice, and let $m_1, \dots, m_r$ be an ordering of the coatoms of L. This ordering satisfies condition (S) of Definition 2.12 if and only if it satisfies the following condition.*

(T) *For all $1 \leq i < j \leq r$ there exists some $k < j$ such that $m_i \wedge m_j \leq m_k \wedge m_j$ and $m_k \wedge m_j$ is a coatom of $[\hat{0}, m_j]$.*

The next result, which follows immediately from Theorems 5.8 and 5.11 of [2], shows that in order to prove that G is shellable it suffices to show that $\mathcal{L}(G)$ admits a recursive coatom ordering.

Theorem 2.14 (Björner-Wachs)**.** *Let P be a finite poset with a unique minimum element $\hat{0}$ and a unique maximum element $\hat{1}$. If P admits a recursive coatom ordering then P is shellable.*

3. Shellable groups are solvable

In this section it will be shown that if a finite group G is shellable then G must be solvable. This result will be achieved by reducing the problem to the examination of the minimal simple groups. Recall that a minimal simple group is a nonabelian finite simple group all of whose proper subgroups are solvable. The reduction is achieved by applying an elementary result of Björner on shellable posets. Recall that a section of a finite group G is a quotient group of a subgroup of G. If G is a nonsolvable finite group then some section of G is a minimal simple group.

Proposition 3.1. *If a finite group G is shellable then every section of G is shellable.*

Proof. Proposition 4.2 of [1] says that if a graded poset P is shellable then every interval in P is shellable. It is straightforward to adapt the proof of this proposition to the nongraded case. Now if $N \trianglelefteq H \leq G$ then the interval $[N, H]$ in $\mathcal{L}(G)$ is isomorphic to $\mathcal{L}(H/N)$ and our proposition follows immediately. □

The goal of this section is achieved by proving the following theorem.

Theorem 3.2. *A minimal finite simple group is not shellable.*

The minimal finite simple groups were determined by Thompson, well before the classification of finite simple groups was achieved. A list is given below.

Theorem 3.3 ([15], Corollary 1)**.** *A finite simple group all of whose proper subgroups are solvable is isomorphic to one of the groups listed below.*

(A) $L_2(p)$ *with* $p \geq 5$ *prime and* $p \not\equiv 1, 4 \bmod 5$.
(B) $L_2(2^p)$ *with* p *prime.*
(C) $L_2(3^p)$ *with* $p \geq 3$ *prime.*
(D) $Sz(2^p)$ *with* $p \geq 3$ *prime.*
(E) $SL_3(3)$.

Note that $L_2(4) \cong L_2(5) \cong A_5$, where A_5 is the alternating group on five symbols. The structures of the subgroup lattices of the minimal simple groups are well-understood, and Theorem 3.2 will be proved by careful examination of each of the five classes of groups given above. We begin by determining all nontrivial subgroups of G which are intersections of maximal subgroups when G is one of the groups described in cases (A) through (D). In what follows, Z_n denotes a cyclic group of order n and D_n denotes a dihedral group of order n. For a prime p, E_{p^a} denotes an elementary abelian group of order p^a. For any groups X and Y, $X.Y$ denotes a split extension of a group isomorphic to X by a group of automorphisms of X isomorphic to Y. Also, $a.b$ will denote $X.Y$ if $X \cong Z_a$ and $Y \cong Z_b$. S_n and A_n denote the symmetric and alternating groups on n letters, respectively.

For a prime power q, all subgroups of $L_2(q)$ are listed in [4]. Those subgroups of $L_2(q)$ which are intersections of maximal subgroups are determined in [7]. The results when $L_2(q)$ is a minimal simple group are as follows.

Lemma 3.4. *Let $p > 7$ be a prime such that $L = L_2(p)$ is a minimal simple group. The maximal subgroups of L are those on the following list.*

- *One conjugacy class of* $\frac{|L|}{p-1}$ D_{p-1}.
- *One class of* $\frac{|L|}{p+1}$ D_{p+1}.
- *One class of* $p+1$ $p.\frac{p-1}{2}$.

- *Two classes of* $\frac{|L|}{24}$ S_4*, if* $p \equiv 1, 7 \bmod 8$.
- *One class of* $\frac{|L|}{12}$ A_4*, if* $p \equiv 3, 5 \bmod 8$.

Furthermore, the nontrivial nonmaximal subgroups of L *which are intersections of maximal subgroups are those on the following list.*

- *One or two classes containing a total of* $\frac{|L|}{8}$ D_8*, if* $p \equiv 1, 7 \bmod 8$.
- *One or two classes containing a total of* $\frac{|L|}{6}$ S_3*, if* $p \equiv 1, 7 \bmod 8$.
- *Two classes of* $\frac{|L|}{24}$ D_4*, if* $p \equiv 1, 7 \bmod 8$.
- *One class of* $\frac{|L|}{12}$ D_4*, if* $p \equiv 3, 5 \bmod 8$.
- *One class of* $\frac{|L|}{p-1}$ $Z_{\frac{p-1}{2}}$.
- *One class of* $\frac{|L|}{v}$ Z_3*, where* $v \in \{p+1, p-1\}$ *and* $3|v$.
- *One class of* $\frac{|L|}{t}$ Z_2*, where* $t \in \{p+1, p-1\}$ *and* $4|t$.

The maximal subgroups of A_5 *are contained in one conjugacy class of five* A_4*, one class of six* D_{10} *and one class of ten* D_6*. The nontrivial nonmaximal subgroups of* A_5 *which are intersections of maximal subgroups are contained in one conjugacy class of ten* Z_3 *and one class of fifteen* Z_2*.*

The maximal subgroups of $L_2(7)$ *are contained in two conjugacy classes each containing seven* S_4 *and one class of eight* 7.3*. The nontrivial nonmaximal subgroups of* $L_2(7)$ *which are intersections of maximal subgroups are contained in one conjugacy class of twenty one* D_8*, one class of twenty eight* D_6*, two classes each containing seven* D_4*, one class containing twenty eight* Z_3 *and one class containing twenty one* Z_2*.*

Lemma 3.5. *Let* p *be an odd prime and set* $q = 2^p$*. The maximal subgroups of* $L = L_2(q)$ *are those on the following list.*

- *One conjugacy class of* $\frac{|L|}{2(q-1)}$ $D_{2(q-1)}$.
- *One class of* $\frac{|L|}{2(q+1)}$ $D_{2(q+1)}$.
- *One class of* $q+1$ $E_{2^p}.Z_{q-1}$.

Furthermore, the nontrivial nonmaximal subgroups of L *which are intersections of maximal subgroups are those on the following list.*

- *One class of* $\frac{|L|}{2(q-1)}$ Z_{q-1}.
- *One class of* $\frac{|L|}{q}$ Z_2.

Lemma 3.6. *Let* p *be an odd prime and set* $q = 3^p$*. The maximal subgroups of* $L = L_2(q)$ *are those on the following list.*

- *One conjugacy class of* $\frac{|L|}{(q-1)}$ D_{q-1}.
- *One class of* $\frac{|L|}{(q+1)}$ D_{q+1}.
- *One class of* $q+1$ $E_{3^p}.Z_{\frac{q-1}{2}}$.
- *One class of* $\frac{|L|}{12}$ A_4.

Furthermore, the nontrivial nonmaximal subgroups of L *which are intersections of maximal subgroups are those on the following list.*

- *One class of* $\frac{|L|}{12}$ D_4.
- *One class of* $\frac{|L|}{(q-1)}$ $Z_{\frac{q-1}{2}}$.
- *Two classes of* $\frac{|L|}{q}$ Z_3.

- *One class of* $\frac{|L|}{q+1}$ Z_2.

The maximal subgroups of the Suzuki groups are determined in [14]. Using further information from [14], one can determine the nonmaximal subgroups of $Sz(2^p)$ which are intersections of maximal subgroups. The results are stated below.

Lemma 3.7. *Let p be an odd prime. Set $q = 2^p$ and $r = \sqrt{2q}$. The maximal subgroups of $G = Sz(q)$ are those on the following list.*

- *One conjugacy class of* $\frac{|G|}{q^2(q-1)}$ $S.Z_{q-1}$, *where* $S \in Syl_2(G)$.
- *One class of* $\frac{|G|}{2(q-1)}$ $D_{2(q-1)}$.
- *One class of* $\frac{|G|}{4(q+r+1)}$ $(q + r + 1).4$.
- *One class of* $\frac{|G|}{4(q-r+1)}$ $(q - r + 1).4$.

Furthermore, the nontrivial nonmaximal subgroups of G which are intersections of maximal subgroups are those on the following list.

- *One class of* $\frac{|G|}{2(q-1)}$ Z_{q-1}.
- *One class of* $\frac{|G|}{2q}$ Z_4.
- *One class of* $\frac{|G|}{q^2}$ Z_2.

We will now use the four lemmas just stated to determine the homotopy type of $\overline{\mathcal{L}(G)}$ in cases (A) through (D) of Theorem 3.3 except in the case $G = L_2(p)$ with $p \equiv 1, 7 \bmod 8$.

Lemma 3.8. *Let G be a minimal simple group isomorphic to one of $L_2(p)$ with $p \equiv 3, 5 \bmod 8$, $L_2(2^p)$, $L_2(3^p)$ or $Sz(2^p)$. Then $\overline{\mathcal{L}(G)}$ has the homotopy type of a wedge of $|G|$ 1-spheres.*

Proof. In each of the given cases, it is known that $\tilde{\chi}(\Delta(\overline{\mathcal{L}(G)})) = -|G|$ (see [9] for $L_2(p)$, [6] or [7] for $L_2(2^p)$, [7] or [12] for $L_2(3^p)$ and [12] for $Sz(2^p)$). Thus it is sufficient to show that $\Delta(\overline{\mathcal{L}(G)})$ is connected and homotopy equivalent to a 1-dimensional complex. To show that $\Delta(\overline{\mathcal{L}(G)})$ is connected, it suffices to show that if X and Y are maximal subgroups of G then there is a path in the 1-skeleton of $\Delta(\overline{\mathcal{L}(G)})$ between (the vertices corresponding to) X and Y. First note that if A and B are subgroups of G of order two then $\langle A, B\rangle$ is proper dihedral subgroup of G. Thus there is a path from A to B. It follows that all maximal subgroups of G having even order are in the same connected component of the 1-skeleton. If $X \leq G$ is a maximal subgroup of odd order, then it follows from the four lemmas above that either $G = L_2(p)$ with $p \equiv 3 \bmod 4$ and $X = p.\frac{p-1}{2}$ or $G = L_2(3^p)$ and $X = E_{3^p}.Z_{\frac{3^p-1}{2}}$. In either case, there exists a dihedral maximal subgroup $D < G$ such that $D \cap X \neq 1$. Thus there is a path from X to D, and it follows that the 1-skeleton is connected.

Let $\mathcal{P}$ be the subposet of $\overline{\mathcal{L}(G)}$ consisting of those nontrivial proper subgroups of G which are intersections of maximal subgroups, of course including the maximal subgroups themselves. By Corollary 2.5, $\Delta(\overline{\mathcal{L}(G)})$ and $\Delta(\mathcal{P})$ have the same homotopy type, so it suffices to show that $\Delta(\mathcal{P})$ is homotopy equivalent to a 1-dimensional complex. By Lemma 3.5, if $G = L_2(2^p)$ then $\Delta(\mathcal{P})$ is already 1-dimensional, so it remains to examine the other three cases.

First assume $G = Sz(2^p)$. It is shown in [14] that G has one conjugacy class of involutions, so by Lemma 3.7 every subgroup of G having order two is in $\mathcal{P}$. It follows that every element of $\mathcal{P}$ isomorphic to Z_4 covers a unique element of $\mathcal{P}$. Let $\mathcal{P}_0$ be the poset obtained by removing all subgroups isomorphic to Z_4. By Corollary 2.4, $\Delta(\mathcal{P})$ and $\Delta(\mathcal{P}_0)$ have the same homotopy type. By Lemma 3.7, $\Delta(\mathcal{P}_0)$ is 1-dimensional.

Finally, assume $G = L_2(q)$ where either q is prime and satisfies the conditions of the lemma or $q = 3^p$ with p an odd prime. By Lemmas 3.4 and 3.6, every facet in $\Delta(\mathcal{P})$ has dimension one or two. Furthermore, if $\sigma = X < Y < Z$ is a 2-simplex in $\Delta(\mathcal{P})$ then exactly one of the following cases holds.

1. $X \cong Z_2$, $Y \cong D_4$ and $Z \cong A_4$.
2. $X \cong Z_3$, $Y \cong Z_{\frac{q-1}{2}}$ $Z \cong D_{q-1}$, q is prime and $q \equiv 1 \bmod 3$.
3. $X \cong Z_2$, $Y \cong Z_{\frac{q-1}{2}}$, $Z \cong q.\frac{q-1}{2}$, q is prime and $q \equiv 1 \bmod 4$.
4. $X \cong Z_2$, $Y \cong D_4$ and $Z \cong D_n$, where $n \in \{q-1, q+1\}$ with $4|n$, and X is not the center of Z.
5. $X \cong Z_2$, $Y \cong Z_{\frac{q-1}{2}}$ $Z \cong D_{q-1}$, q is prime and $q \equiv 1 \bmod 4$.
6. $X \cong Z_2$, $Y \cong D_4$ and $Z \cong D_n$, where $n \in \{q-1, q+1\}$ with $4|n$, and X is the center of Z.

In the first four cases above, σ is the unique facet of $\Delta(\mathcal{P})$ containing $\tau(\sigma) = X < Z$. Let Δ_0 be the complex obtained by removing from $\Delta(\mathcal{P})$ all pairs $\sigma, \tau(\sigma)$ where σ is one of the 2-simplices described in the first four cases above. By Lemma 2.6, $\Delta(\mathcal{P})$ and Δ_0 are homotopy equivalent. Now let σ be as in the fifth or sixth case above. Then σ is the unique facet in Δ_0 containing $\tau(\sigma) = Y < Z$. Let Δ_1 be the complex obtained by removing all pairs $\sigma, \tau(\sigma)$ where σ is one of the 2-simplices described in the fifth or sixth case above from Δ_0. Again by Lemma 2.6, Δ_0 is homotopy equivalent to the 1-dimensional complex Δ_1 and the lemma follows. $\square$

Corollary 3.9. *Let G be a minimal simple group isomorphic to one of $L_2(p)$ with $p \equiv 3, 5 \bmod 8$ and $p > 5$, $L_2(2^p)$ with p an odd prime, $L_2(3^p)$ or $Sz(2^p)$. Then $\Delta(\overline{\mathcal{L}(G)})$ is not shellable.*

Proof. By Theorem 2.9 and Lemma 3.8, if $\Delta(\overline{\mathcal{L}(G)})$ is shellable then it contains at least $|G|$ 1-dimensional facets. We will examine each possibility for G using Lemmas 3.4, 3.5, 3.6 and 3.7, and show that this condition cannot hold. Note that a 1-dimensional facet in $\Delta(\overline{\mathcal{L}(G)})$ is a chain $X < Y$ such that Y is a maximal subgroup of G, X is a maximal subgroup of Y and $|X|$ is prime.

If $G = L_2(p)$ with $p \equiv 3, 5 \bmod 8$, then $\Delta(\overline{\mathcal{L}(G)})$ contains $\frac{|G|}{3}$ 1-dimensional facets $X < Y$ with $X \cong Z_3$ and $Y \cong A_4$. There are no other 1-dimensional facets in $\Delta(\overline{\mathcal{L}(G)})$ unless $\frac{p-1}{2}$ or $\frac{p+1}{2}$ is prime. Since $p > 5$, at most one of these two numbers can be prime. If $\frac{p-1}{2}$ is prime then the remaining 1-dimensional facets in $\Delta(\overline{\mathcal{L}(G)})$ are $\frac{|G|}{2}$ chains $X < Y$ with $X \cong Z_2$ and $Y \cong D_{p-1}$, $\frac{|G|}{p-1}$ chains $X < Y$ with $X \cong Z_{\frac{p-1}{2}}$ and $Y \cong D_{p-1}$, $\frac{2|G|}{p-1}$ chains $X < Y$ with $X \cong Z_{\frac{p-1}{2}}$ and $Y \cong p.\frac{p-1}{2}$ and $\frac{2|G|}{p(p-1)}$ chains $X < Y$ with $X \cong Z_p$ and $Y \cong p.\frac{p-1}{2}$. It follws that if there are at least $|G|$ 1-facets then $p < 20$. However, the only primes $p < 20$ with $p \equiv 3, 5 \bmod 8$ and $\frac{p-1}{2}$ prime are 5 and 11, and $L_2(11)$ contains A_5 and is therefore not a minimal simple group. If $\frac{p+1}{2}$ is prime then the remaining 1-dimensional facets in $\Delta(\overline{\mathcal{L}(G)})$

are $\frac{|G|}{2}$ chains $X < Y$ with $X \cong Z_2$ and $Y \cong D_{p+1}$ and $\frac{|G|}{p+1}$ chains with $X \cong Z_{\frac{p+1}{2}}$ and $Y \cong D_{p+1}$. Since $p > 5$, there are less than $|G|$ 1-dimensional facets.

If $G = L_2(q)$ where $q = 2^p$ with p an odd prime then $\Delta(\overline{\mathcal{L}(G)})$ contains no 1-dimensional facets unless $q-1$ is prime. In that case, the 1-dimensional facets in $\Delta(\overline{\mathcal{L}(G)})$ are $\frac{|G|}{2}$ chains $X < Y$ with $X \cong Z_2$ and $Y \cong D_{2(q-1)}$, $\frac{|G|}{2(q-1)}$ chains $X < Y$ with $X \cong Z_{q-1}$ and $Y \cong D_{2(q-1)}$ and $\frac{|G|}{q-1}$ chains $X < Y$ with $X \cong Z_{q-1}$ and $Y \cong E_{2^p}.Z_{q-1}$. The total number of 1-dimensional facets in this case is less than $|G|$.

If $G = L_2(q)$ where $q = 3^p$ with p an odd prime then $\Delta(\overline{\mathcal{L}(G)})$ contains $\frac{|G|}{3}$ facets $X < Y$ with $X \cong Z_3$ and $Y \cong A_4$, and no other 1-dimensional facets unless $\frac{q-1}{2}$ is prime. If $\frac{q-1}{2}$ is prime, then the additional 1-dimensional facets of $\Delta(\overline{\mathcal{L}(G)})$ are $\frac{|G|}{2}$ chains $X < Y$ with $X \cong Z_2$ and $Y \cong D_{q-1}$, $\frac{|G|}{q-1}$ chains $X < Y$ with $X \cong Z_{\frac{q-1}{2}}$ and $Y \cong D_{q-1}$ and $\frac{2|G|}{q-1}$ chains $X < Y$ with $X \cong Z_{\frac{q-1}{2}}$ and $Y \cong E_{3^p}.Z_{\frac{q-1}{2}}$. Since $q \geq 27$, there are less than $|G|$ 1-dimensional facets.

If $G = Sz(q)$ where $q = 2^p$ with p an odd prime then $\Delta(\overline{\mathcal{L}(G)})$ has no 1-dimensional facets unless $q-1$ is prime. If $q-1$ is prime then the 1-dimensional facets of $\Delta(\overline{\mathcal{L}(G)})$ are $\frac{|G|}{2}$ chains $X < Y$ with $X \cong Z_2$ and $Y \cong D_{2(q-1)}$ and $\frac{|G|}{2(q-1)}$ chains $X < Y$ with $X \cong Z_{q-1}$ and $Y \cong D_{2(q-1)}$. Here we have used the fact, stated in [14], that if $S \in Syl_2(G)$ then S is not abelian. Thus S has a characteristic nontrivial proper subgroup and $S.Z_{q-1}$ has no maximal subgroup of order $q-1$. In any case, $\Delta(\overline{\mathcal{L}(G)})$ has less than $|G|$ 1-dimensional facets and we are done. □

Lemma 3.10. *If $G = A_5$ then $\Delta(\overline{\mathcal{L}(G)})$ is not shellable.*

Proof. Write Δ for $\Delta(\overline{\mathcal{L}(G)})$. If Δ is shellable then by Theorem 2.11 so is $\Delta^{(2,2)}$. By Lemma 3.4, the facets of $\Delta^{(2,2)}$ are 15 chains $X < Y < Z$ with $X \cong Z_2$, $Y \cong D_4$ and $Z \cong A_4$. If σ and τ are two such facets then $|\sigma \cap \tau| \leq 1$. It follows immediately from Definition 1.1 than $\Delta^{(2,2)}$ is not shellable. □

It remains to examine the minimal simple groups $L_2(p)$ with $p \equiv 1, 7 \bmod 8$, along with $SL_3(3)$.

Lemma 3.11. *Let $G = L_2(p)$ be a minimal simple group with $p \equiv 1, 7 \bmod 8$. Then $\tilde{H}_1(\Delta(\overline{\mathcal{L}(G)})) \neq 0$.*

Proof. Let $\mathcal{P}$ be the subposet of $\overline{\mathcal{L}(G)}$ consisting of those groups which are intersections of maximal subgroups of G. By Corollary 2.5, it is sufficient to show that $\tilde{H}_1(\Delta(\mathcal{P})) \neq 0$. Set $r = \frac{p-1}{2}$. By Lemma 3.4, $\Delta(\mathcal{P})$ contains 1-simplices $X < Y$ with $X \cong Z_r$ and $Y \cong p.r$. Furthermore, any facet of $\Delta(\mathcal{P})$ properly containing one of these 1-simplices is of the form $\sigma = W < X < Y$ with $W \cong Z_a$, where $a \in \{2, 3\}$ and $a|r$. For such a facet σ, let $\tau(\sigma) = W < Y$. Then σ is the unique facet of $\Delta(\mathcal{P})$ containing $\tau(\sigma)$. Let Δ_0 be the complex obtained by removing all pairs $\sigma, \tau(\sigma)$ described above from $\Delta(\mathcal{P})$. By Lemma 2.6, $\Delta(\mathcal{P})$ and Δ_0 have the same homotopy type, so it suffices to show that $\tilde{H}_1(\Delta_0) \neq 0$. Let Δ_1 be the subcomplex of Δ_0 whose facets are all chains $X < Y$ with $X \cong Z_r$ and $Y \cong p.r$. Then Δ_1 is a graph with $p + 1 + \frac{p(p+1)}{2}$ vertices and $p(p+1)$ edges. Since $p > 2$, Δ_1 has more edges than vertices, and is not a forest. Therefore, $\tilde{H}_1(\Delta_1) \neq 0$. Since every 1-simplex in Δ_1 is a facet in Δ_0, $\tilde{H}_1(\Delta_0) \neq 0$ and we are done. □

Corollary 3.12. *Let $G = L_2(p)$ be a minimal simple group with $p \equiv 1, 7 \bmod 8$. Then $\Delta(\overline{\mathcal{L}(G)})$ is not shellable.*

Proof. By Lemma 3.11 and Theorem 2.9(2), if $\Delta(\overline{\mathcal{L}(G)})$ is shellable then it contains some 1-dimensional facets. By Lemma 3.4, $\Delta(\overline{\mathcal{L}(G)})$ contains 1-dimensional facets if and only if either $\frac{p-1}{2}$ or $\frac{p+1}{2}$ is prime. We will examine both of these possibilities. From now on, we write Δ for $\Delta(\overline{\mathcal{L}(G)})$. Let d be the dimension of the largest facet in Δ, so $d = l(G) - 1$.

First assume that $\frac{p+1}{2}$ is prime, and set $r = \frac{p-1}{2}$. Note that in this case, $p \equiv 1 \bmod 8$, so $2|r$. By Lemma 3.4, $\Delta^{(2,d)}$ is the order complex of the ideal in $\overline{\mathcal{L}(G)}$ generated by maximal subgroups of G isomorphic to D_{2r}, $p.r$ or S_4. Let $\mathcal{P}$ be the subposet of $\overline{\mathcal{L}(G)}$ consisting of those subgroups of G which are intersections of these maximal subgroups. Then $\Delta^{(2,d)}$ and $\Delta(\mathcal{P})$ have the same homotopy type by Corollary 2.5, and $\Delta(\mathcal{P})$ contains the chains $X < Y$ with $X \cong Z_r$ and $Y \cong p.r$ described in Lemma 3.11. The same argument used in the proof of that Lemma shows that $\tilde{H}_1(\Delta^{(2,d)}) \neq 0$. It now follows from Theorem 2.9(2) and the fact that $\Delta^{(2,d)}$ has no 1-dimensional facets that $\Delta^{(2,d)}$ is not shellable. By Theorem 2.11, Δ is not shellable.

Now assume that $\frac{p-1}{2}$ is prime, and set $s = \frac{p+1}{2}$. Note that in this case, $p \equiv 7 \bmod 8$, so $8|2s$. It follows that exactly one of the following three cases holds.

1. $p + 1 = 2s$ has at least five prime factors, counting multiplicities.
2. $3|p - 1$, so $p = 7$.
3. $p + 1 = 24$, so $p = 23$.

If $2s$ has at least five prime factors, then by Lemma 3.4 $\Delta^{(d,d)}$ is the order complex of the ideal in $\overline{\mathcal{L}(G)}$ generated by the maximal subgroups of G isomorphic to D_{2s}. Let $\mathcal{P}$ be the subposet of $\overline{\mathcal{L}(G)}$ consisting of those groups which are intersections of these maximal subgroups, so $\Delta^{(d,d)}$ and $\Delta(\mathcal{P})$ have the same homotopy type by Corollary 2.5. By Lemma 3.4, $\mathcal{P}$ consists of those subgroups of G isomorphic to Z_2,D_4 or D_{2s}. Let $\mathcal{Q}$ be the subposet of $\mathcal{P}$ consisting of those subgroups of G isomorphic to D_4 or Z_2, and let $i : \mathcal{Q} \hookrightarrow \mathcal{P}$ be the identity embedding. Note that for each $X \in \mathcal{P}$, i_X is the poset of nontrivial elementary abelian 2-subgroups of X. Also, note that $O_2(X) \neq 1$, that is, each $X \in \mathcal{P}$ contains a nontrivial normal 2-subgroup. By [11], Lemma 2.2, i_X is contractible for all $X \in \mathcal{P}$. Thus by Proposition 2.3, $\Delta(\mathcal{P})$ and $\Delta(\mathcal{Q})$ have the same homotopy type. Now $\Delta(\mathcal{Q})$ is a graph with $\frac{|G|}{12} + \frac{|G|}{2s}$ vertices and $\frac{|G|}{4}$ edges. Since there are more edges than vertices, $\Delta(\mathcal{Q})$ is not a forest and $\tilde{H}_1(\Delta(\mathcal{Q})) \neq 0$. Thus $\tilde{H}_1(\Delta^{(d,d)}) \neq 0$, and since $\Delta^{(d,d)}$ has no 1-dimensional facets, it is not shellable by Theorem 2.9(2). Now Δ is not shellable by Theorem 2.11.

If $p = 7$ then by Lemma 3.4 $\Delta^{(3,3)}$ is the order complex of the subposet $\mathcal{P}$ of $\overline{\mathcal{L}(G)}$ consisting of those subgroups of G isomorphic to Z_2,Z_4,D_4,D_8,A_4 or S_4. Note that $O_2(X) \neq 1$ for each $X \in \mathcal{P}$. Now as in the above paragraph, we see that $\Delta^{(3,3)}$ and $\Delta(\mathcal{Q})$ have the same homotopy type, where $\mathcal{Q}$ is the subposet of $\overline{\mathcal{L}(G)}$ consisting of those subgroups of G isomorphic to Z_2 or D_4, and that $\tilde{H}_1(\Delta(\mathcal{P})) \neq 0$. As above, it follows that Δ is not shellable.

If $p = 23$ the situation is more complicated, but it can be shown that Δ is not shellable either by using Lemma 2.6 to show that $\Delta^{(3,3)}$ has the same homotopy type as a noncontractible 2-dimensional complex, or by calculating that $h_{4,3} = -3,040$ and applying Theorem 2.9(1). This last calculation can be done by hand. $\square$

Lemma 3.13. *If $G = SL_3(3)$ then $\Delta(\overline{\mathcal{L}(G)})$ is not shellable.*

Proof. Write Δ for $\Delta(\overline{\mathcal{L}(G)})$, and let $d = l(G) - 1$ be the dimension of the largest facet of Δ. We will see that $\Delta^{(d,d)}$ is not shellable and apply Theorem 2.11. Let V be a 3-dimensional vector space over $\mathbf{F}_3$ on which G acts naturally. It is known that the only subgroups of G which are of prime index in G are the stabilizers in G of the nontrivial proper subspaces of V (see for example [3]). For any finite solvable group H, $l(H) + 2$ is equal to the number of prime factors of $|H|$, counting multiplicities. It follows that the facets of $\Delta^{(d,d)}$ are chains $X_0 < X_1 < \ldots < X_d$ such that X_d is a subspace stabilizer and $[X_i : X_{i-1}]$ is prime for $1 \leq i \leq d$.

It is known that G acts 2-transitively on the set of 13 1-spaces in V and on the set of 13 2-spaces in V. Let $M < G$ be the stabilizer of a fixed 1-space $W < V$. If K is the stabilizer of another 1-space, then by 2-transitivity $[M : K \cap M] = 12$. Also by 2-transitivity, M acts transitively on the set of four 2-spaces in V which contain W, so if K is the stabilizer of such a 2-space then $[M : K \cap M] = 4$. It can be shown using elementary linear algebra that M also acts transitively on the set of nine 2-spaces in V which do not contain W, so if K is the stabilizer of such a 2-space the $[M : K \cap M] = 9$. The stabilizers in G of 1-spaces from V are conjugate in $Aut(G)$ with the stabilizers of 2-spaces. We now see that if X and Y are distinct subspace stabilizers then $[X : X \cap Y] \in \{4, 9, 12\}$. In particular, $X \cap Y$ is not of prime index in X. Thus if $X_0 < \ldots < X_{d-1} < X_d$ and $Y_0 < \ldots < Y_{d-1} < Y_d$ are facets of $\Delta^{(d,d)}$ with $X_d \neq Y_d$ then $X_{d-1} \neq Y_{d-1}$. It now follows from Definition 1.1 that $\Delta^{(d,d)}$ is not shellable. Indeed, if $\sigma_1, \ldots, \sigma_r$ is a list of the facets, let i be the smallest index such that the maximum element of σ_i is not the maximum element of σ_1. Then $|\sigma_i \cap \sigma_j| < |\sigma_i| - 1$ for all $j < i$, and this list is not a shelling. $\square$

The proof of Lemma 3.13 completes the proof of Theorem 3.2. Note that many of the results in this section can be obtained in the same manner as Lemma 3.13, that is, by examining intersections of maximal subgroups of a minimal simple group G and showing that the complex $\Delta^{(d,d)}$ cannot be shellable. The proofs used here were chosen because they give some insight into the homotopy type and homology of $\Delta(\overline{\mathcal{L}(G)})$. We close this section by noting one further result.

Proposition 3.14. *Let G be a minimal simple group. Then $\tilde{H}_1(\Delta(\overline{\mathcal{L}(G)})) \neq 0$.*

Proof. Given Lemmas 3.8 and 3.11, it is now sufficient to prove the claim when $G = SL_3(3)$. Using information given in [3], we see that $\Delta(\overline{\mathcal{L}(G)})$ contains facets of the form $X < Y$ with $X \cong Z_3$ and $Y \cong 13.3$. Let Δ_0 be the subcomplex generated by these facets. Again using [3], we see that Δ_0 is a graph with $\frac{|G|}{18} + \frac{|G|}{39}$ vertices and $\frac{|G|}{3}$ edges. Thus Δ_0 is not a forest, so $\tilde{H}_1(\Delta_0) \neq 0$ and we are done. $\square$

4. Solvable groups are shellable

In this section, it will be shown that if G is a solvable finite group then $\mathcal{L}(G)$ admits a recursive coatom ordering. It then follows from Theorem 2.14 that $\Delta(\overline{\mathcal{L}(G)})$ is shellable, and this completes the proof of Theorem 1.4. The reader who has read [16] should be aware that the proposed counterexample to the claim that all solvable finite groups are shellable given in that paper is incorrect, as a dihedral subgroup of order ten has been omitted from the subgroup lattice of the given group.

We will need some facts about the intersections of maximal subgroups of solvable groups. Recall that if H is a subgroup of a finite group G then $Core_G(H)$ is the largest normal subgroup of G contained in H.

Theorem 4.1 (Ore). *Let L and M be maximal subgroups of a finite solvable group G. Then L and M are conjugate in G if and only if $Core_G(L) = Core_G(M)$.*

Proof. See Theorem (16.1) of section A of [5]. □

Given the above theorem, the partial order described in the following definition is well-defined.

Definition 4.2. Let G be a finite solvable group. For $H \leq G$, H^G will denote the conjugacy class of H in G. Define a partial order $\preceq$ on the conjugacy classes of maximal subgroups of G by $L^G \preceq M^G$ if and only if $Core_G(L) \leq Core_G(M)$.

Theorem 4.3. *Let L and M be maximal subgroups of a finite solvable group G which are not conjugate in G. If $M^G \not\preceq L^G$, then $L \cap M$ is a maximal subgroup of L.*

Proof. This is Theorem (16.6) of Section A of [5]. □

Lemma 4.4. *Let M, M' be conjugate maximal subgroups of a finite solvable group G, with $M \neq M'$. Let J be any proper subgroup of M containing $M \cap M'$. Then there exists some maximal subgroup $L < G$ such that $J \leq L$ and $M^G \prec L^G$.*

Proof. Let $C = Core_G(M) = Core_G(M')$. There exists a unique subgroup $K \leq G$ such that $C < K \trianglelefteq G$, $MK = M'K = G$ and $M \cap K = M' \cap K = C$ (see (15.2) of Section A of [5]). An order argument shows that $JK < G$. Let L be any maximal subgroup of G containing JK. Since $K \leq L$, we have $C < K \leq Core_G(L)$, so $M^G \prec L^G$. □

Definition 4.5. Let G be a finite solvable group. An SC-list for G is an ordering $\mathcal{M}$ of the maximal subgroups of G with the following properties.

(1) If L, M are maximal subgroups of G which are not conjugate in G and L appears before M in $\mathcal{M}$, then every G-conjugate of L appears before M in $\mathcal{M}$.

(2) If L, M are maximal subgroups of G and $M^G \prec L^G$ then L appears before M in $\mathcal{M}$.

Note that condition (1) of Definition 4.5 requires that the conjugates of any maximal subgroup of G form a segment in any SC-list for G. Also, it follows immediately from the fact that the partial order $\preceq$ has a linear extension that an SC-list for G exists for any finite solvable group G. Indeed, one may take the reverse of a given linear extension and arbitrarily list all of the elements of each conjugacy class.

Lemma 4.6. *Let G be a finite solvable group and let $\mathcal{M}$ be an SC-list for G. Then $\mathcal{M}$ satisfies condition (T) of Proposition 2.13.*

Proof. We must show that if L appears before M in $\mathcal{M}$ then there exists some K appearing before M in $\mathcal{M}$ such that $L \cap M \leq K \cap M$ and $K \cap M$ is maximal in M. First assume that L and M are not conjugate in G. Since L appears before M in G, $L^G \not\preceq M^G$ and $L \cap M$ is maximal in M by Theorem 4.3. Thus we can take $K = L$. Now assume L and M are conjugate in G. Let J be any maximal subgroup of M containing $L \cap M$. By Lemma 4.4, there exists some maximal $K < G$ such that $J \leq K$ and $M^G \prec K^G$. By condition (2) of Definition 4.5, K appears before M in $\mathcal{M}$. Since J is maximal in M, $K \cap M = J$. □

We will now see that we can choose SC-lists for all nontrivial subgroups of a finite group G in a manner that satisfies condition (R) of definition 2.12.

Lemma 4.7. *Let G be a finite solvable group and let $\mathcal{M} = M_1, \ldots, M_r$ be an SC-list for G. Then for any $j \in [r]$, the set of maximal subgroups of M_j of the form $M_i \cap M_j$ with $i < j$ is a union of M_j-conjugacy classes.*

Proof. We must show that if $H = M_i \cap M_j$ with $i < j$ is maximal in M_j, then for every $x \in M_j$ there exists some $k < j$ such that $H^x = M_k \cap M_j$. If M_i is conjugate to M_j in G then by Lemma 4.4 there exists some M_l such that $H < M_l$ and $M_j^G \prec M_l^G$. Since M_l is not conjugate to M_j in G and $\mathcal{M}$ is an SC-list, $l < j$. Since H is maximal in M_j, $M_l \cap M_j = H$. Therefore, we may assume without loss of generality that M_i and M_j are not conjugate in G. Now for any $x \in M_j$ we have

$$H^x = (M_i \cap M_j)^x = M_i^x \cap M_j.$$

Since $\mathcal{M}$ is an SC-list and M_i appears before M_j in $\mathcal{M}$ with M_i and M_j not conjugate in G, M_i^x appears before M_j in $\mathcal{M}$. □

Lemma 4.8. *Let G be a finite solvable group and let $\mathcal{M} = M_1, \ldots, M_r$ be an SC-list for G. Fix $j \in [r]$. Let $H = M_i \cap M_j$ be a maximal subgroup of M_j with $i < j$. Let X be a maximal subgroup of M_j with $Core_{M_j}(H) \leq Core_{M_j}(X)$. Then there is some $l < j$ such that $X = M_l \cap M_j$.*

Proof. As in the proof of Lemma 4.7, we may assume that M_i and M_j are not conjugate in G. Now

$$Core_G(M_i) \cap M_j \leq Core_{M_j}(H) \leq Core_{M_j}(X) \leq X,$$

so $Core_G(M_i) \cap M_j = Core_G(M_i) \cap X$. An order argument shows that $X Core_G(M_i) < G$. Let M_l be any maximal subgroup of G containing $X Core_G(M_i)$. Since X is maximal in M_j, $M_l \cap M_j = X$. We now have

1. $i < j$,
2. $Core_G(M_i) \leq Core_G(M_l)$,
3. M_i and M_j are not conjugate in G, and
4. $\mathcal{M}$ is an SC-list for G.

It follows that $l < j$. □

Corollary 4.9. *Let G be a finite solvable group and let $\mathcal{M} = M_1, \ldots, M_r$ be an SC-list for G. Then for each $j \in [r]$ there exists an SC-list for M_j in which all maximal subgroups of M_j of the form $M_i \cap M_j$ with $i < j$ appear before all other maximal subgroups of M_j.*

Proof. Fix j and let $\mathcal{J}_1$ be the set of all maximal subgroups of M_j of the form $M_i \cap M_j$ with $i < j$, and let $\mathcal{J}_2$ be the set of all the other maximal subgroups of M_j. By Lemma 4.7, each $\mathcal{J}_i$ is a union of M_j-conjugacy classes. Let $\preceq_j$ be the partial order on conjugacy classes of maximal subgroups of M_j as given in Definition 4.2. Using the restriction of $\preceq_j$ to the M_j-conjugacy classes contained in $\mathcal{J}_i$, we see that for each $i = 1, 2$ there exists a list $\mathcal{R}_i$ of the elements of $\mathcal{J}_i$ satisfying the following conditions.

(a) If $X, Y \in \mathcal{J}_i$ with X and Y not conjugate in M_j and X appears before Y in $\mathcal{R}_i$ then every M_j-conjugate of X appears before Y in $\mathcal{R}_i$.

(b) If $X, Y \in \mathcal{J}_i$ and $X^{M_j} \preceq_j Y^{M_j}$ then Y appears before X in $\mathcal{R}_i$.

Let $\mathcal{R}$ be the list of maximal subgroups of M_j obtained by placing $\mathcal{R}_2$ after $\mathcal{R}_1$. Given the fact that each $\mathcal{R}_i$ satisfies conditions (a) and (b) above, it is sufficient to show that if $Y \in \mathcal{J}_2$ and $X \in \mathcal{J}_1$ then $X \not\preceq_j Y$. This follows immediately from Lemma 4.8. □

Lemma 4.6 and Corollary 4.9 give the following result.

Corollary 4.10. *If G is a finite solvable group then $\mathcal{L}(G)$ admits a recursive coatom ordering.*

By Theorem 2.14, $\mathcal{L}(G)$ is shellable, and he proof of Theorem 1.4 is complete.

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