COMPLEXES OF t-COLORABLE GRAPHS

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Abstract. We study the simplicial complex of t-colorable graphs on n vertices. We prove this complex is homotopy equivalent to a wedge of spheres all of dimension $n(t-1)-\binom{t}{2}-1$ when $t=2$ and when $t \geq n-3$. We show that such a homotopy equivalence does not hold for general t and n.

1. Introduction and statement of results

Recall that a monotone graph property is a collection Γ of graphs on a fixed labeled vertex set V which satisfies the two conditions

(A) If $G \in \Gamma$ and H is obtained from G by removing an edge then $H \in \Gamma$, and
(B) If $G \in \Gamma$ and H is isomorphic to G then $H \in \Gamma$.

From now on we assume that $V=[n]$ for some $n \in \mathbb{N}$. Any collection Γ of graphs on V which satisfies condition (A) determines a simplicial complex (also called Γ) whose vertex set corresponds to the set of edges in the complete graph K_n. That is, a graph $G \in \Gamma$ with $k+1$ edges corresponds to a k-dimensional face whose proper faces correspond to proper subgraphs of G. If Γ also satisfies condition (B) then the action of the symmetric group S_n on V determines an action of S_n on Γ as a group of simplicial automorphisms.

The theory of group actions on topological spaces was successfully applied to the study of monotone graph properties by J. Kahn, M. Saks and D. Sturtevant in their examination of computational complexity (see [KSS]). More recently, certain classes of monotone graph properties have appeared in various areas of mathematics. Complexes of graphs of bounded degree arise in group theory, topology, combinatorics, geometry and commutative algebra (see [Bo, BLVZ, KRW, RR]) and complexes of graphs of bounded connectivity arise in the study of invariants of knots and similar objects (see [BBLSW, Sh1, Sh2, Tu, Va1, Va2, Va3]). The appealing results discovered in the papers mentioned above led us to investigate the topology of other monotone graph properties which have simple graph-theoretic definitions. In this paper we discuss our results on complexes of graphs of bounded chromatic number.

Recall that a graph G is t-colorable if its vertices can be colored with t colors so that no two adjacent vertices receive the same color. The chromatic number $\chi(G)$ is the smallest t such that G is t-colorable. For $1 \leq t \leq n$, the complex of t-colorable graphs on n vertices is

$$\Gamma_n^t := \{G \subseteq K_n : \chi(G) \leq t\}.$$

If $t=1$ then Γ_n^t consists of only the empty graph, while if $t=n$ then Γ_n^t is the full $(\binom{n}{2}-1)$-simplex. The only graph on n vertices which is not $(n-1)$-colorable

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is the complete graph K_n, so Γ_n^{n-1} is the boundary of the $(\binom{n}{2}-1)$-simplex and is therefore homeomorphic with the $(\binom{n}{2}-2)$-sphere. We are able to determine the homotopy type of Γ_n^t when t is one of $2, n-2$ or $n-3$.

To state our result in the case $t = 2$ we need some definitions. Recall that for vertices x, y of G, the distance $d(x, y)$ from x to y is defined by $d(x, y) = \infty$ if x and y are not in the same connected component of G, and $d(x, y)$ is the length, i.e. the number of edges, of the shortest path from x to y in G otherwise. We now define, for any graph $G = ([n], E)$ and $i \in \mathbb{N} \cup \{\infty\}$,

$$D_i(G) := \{x \in [n] \setminus \{1\} : d(1, x) = i\},$$

and

$$md(G) = max\{i \in \mathbb{N} : D_i(G) \neq \emptyset\}.$$

For $1 \leq i \leq md(G)$, define

$$a_i(G) := minD_i(G).$$

Now define $\mathcal{T}(n)$ to be the set of all trees T on vertex set $[n]$ which satisfy the three conditions

- for $1 \leq i < md(T)$, the only element of $D_i(T)$ which has neighbors in $D_{i+1}(T)$ is $a_i(T)$,
- for $1 \leq i < md(T)$, there is some element of $\bigcup_{j \geq i} D_j(T)$ which is larger than $a_i(T)$, and
- $|D_{md(T)}(T)| > 1$.

See Figure 1 for a list of the trees in $\mathcal{T}(5)$.

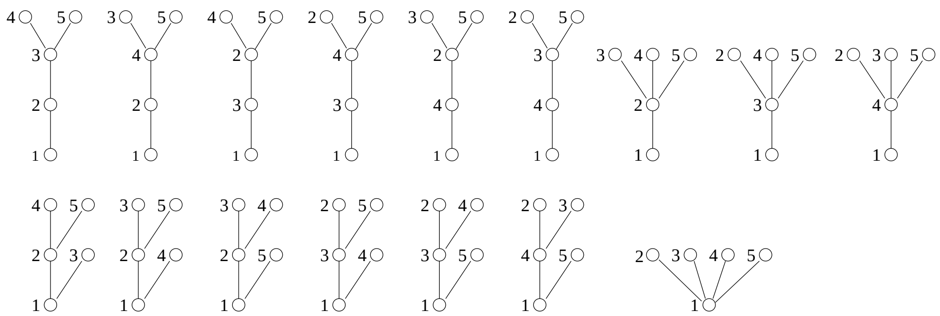

FIGURE 1. The 16 trees in $\mathcal{T}(5)$.

In Section 4, we will prove the following result.

Theorem 1.1. *For all $n \geq 2$ the complex Γ_n^2 has the homotopy type of a wedge of $|\mathcal{T}(n)|$ spheres of dimension $n-2$.*

The fact that Γ_n^2 has the homotopy type of a wedge of spheres of dimension $n-2$ is a special case of a result of M. Chari (see [Ch2]), who showed that if every edge of some graph G is contained in an odd cycle then the complex $\Gamma^2(G)$ of all 2-colorable subgraphs of G has the homotopy type of a wedge of spheres of dimension $|V(G)|-2$. It is also the case that if some edge of G is not contained in an odd cycle then $\Gamma^2(G)$ is collapsible and therefore contractible. Further discussion of the relationship between our result and Chari's appears in Section 4.

Using a result that appears as an exercise in [St], we can obtain a generating function for $|\mathcal{T}(n)|$.

Corollary 1.2. *The generating function*

$$\mathcal{B}(z) := \sum_{n\geq 2} |\mathcal{T}(n)| \frac{z^n}{n!}$$

satisfies

$$\mathcal{B}(z) = -\sqrt{2e^{-z}-1} - z + 1.$$

Proof. By Theorem 1.1, $|\mathcal{T}(n)|$ is equal to the reduced Euler characteristic $\tilde{\chi}(\Gamma_n^2)$. In exercise 5.5 of [St], it is shown that if $b(n,m)$ is the number of bipartite graphs on vertex set $[n]$ with m edges then

$$\sum_{n,m\geq 0} b(n,m)x^n q^m = \left[\sum_{n\geq 0} \left(\sum_{i=0}^{n} (1+q)^{i(n-i)} \binom{n}{i} \right) \frac{x^n}{n!} \right]^{1/2}.$$

Setting $q=-1$ and $x=-z$ completes the proof. $\square$

We now turn to the cases $t = n-2$ and $t = n-3$. In Section 5 we will prove the following results.

Theorem 1.3. *For each $n \geq 3$ the complex Γ_n^{n-2} has the homotopy type of a wedge of $\binom{n-1}{2}$ spheres of dimension $\binom{n}{2} - 4$.*

Theorem 1.4. *For each $n \geq 4$ the complex Γ_n^{n-3} has the homotopy type of a wedge of $\binom{n-1}{3} + 12\binom{n}{5}$ spheres of dimension $\binom{n}{2} - 7$.*

With Theorems 1.1, 1.3 and 1.4 in hand, it is reasonable to conjecture that Γ_n^t has the homotopy type of a wedge of spheres of dimension $n(t-1)-\binom{t}{2}-1$ for all n,t. However, computer calculations show that this conjecture is false. In particular, $\widetilde{H}_{19}(\Gamma_8^4)$ is free of rank one. On the other hand, computer calculations also show that $\widetilde{H}_i(\Gamma_8^4) = 0$ for $i \notin \{17,19\}$ and $\widetilde{H}_{17}(\Gamma_8^4)$ is free of rank $9,396$. So, one can still ask whether Γ_n^t always has nontrivial free homology in dimension $n(t-1)-\binom{t}{2}-1$ and whether this is the smallest dimension in which nontrivial homology appears. In the course of proving Theorems 1.3 and 1.4, we prove a lemma from which the next result follows immediately.

Proposition 1.5. *For all n,t the complex Γ_n^t is $(\lceil \frac{(n-1)(t-1)}{2} \rceil - 1)$-connected.*

Although Theorems 1.1, 1.3 and 1.4 can all be obtained using traditional tools from topological combinatorics such as the Quillen fiber lemma, Alexander duality and the theory of lexicographic shellability (see [Bj]), the proofs we will give all use the discrete Morse theory of R. Forman, which is described in [Fo1]. These proofs, although somewhat technical and complicated, are considerably simpler and more natural than the proofs which use the traditional tools. One of our main purposes for writing this paper is to further demonstrate the power of Forman's theory in the study of monotone graph properties (other evidence appears in [BBLSW, Fo2, Jo, Sh1, Sh2]) and to explain certain techniques which have proven useful when applying the theory.

After giving explicit definitions of some of the objects mentioned in this introduction and introducing some other necessary concepts in Section 2, we discuss discrete Morse theory in Section 3. The proof of Theorem 1.1 and some further discussion of that theorem appears in Section 4, and the proofs of Theorems 1.3 and 1.4 appear in Section 5. Concluding remarks appear in Section 6.

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2. Preliminaries

By a graph $G = (V(G), E(G))$ we mean a loopless graph without multiple edges (equivalently: a one-dimensional simplicial complex) on the vertex set $V(G)$ with edge set $E(G) \subseteq \binom{V(G)}{2}$. Our standard vertex set will be the set $[n] := \{1, 2, \ldots, n\}$. A graph G is called *t-colorable* if there is a function $f : V(G) \to [t]$ such that if $\{x, y\} \in E(G)$ then $f(x) \neq f(y)$. Thus the preimages $\{f^{-1}(i) : i \in [t]\}$ partition $V(G)$ into (at most) t independent sets. Graphs which are 2-colorable are also called bipartite.

Of course, if $G = (V(G), E(G))$ is a graph that is t-colorable for some $t \geq 2$ then for any subset $E' \subseteq E(G)$ the graph $G' = (V(G), E')$ on the same vertex set is also t-colorable. Hence if we fix an n-element vertex set V (e.g. $[n]$) and identify a graph with the set of its edges, then we may regard the set of t-colorable graphs on V as a simplicial complex.

Definition: Γ_n^t is the complex of t-colorable graphs on n vertices ($2 \leq t \leq n$). Its simplices are the subsets $E \subseteq \binom{[n]}{2}$ such that the graph $([n], E)$ is t-colorable.

For a graph G and a vertex v, $G - v$ will denote the graph that is obtained from G by deleting the vertex v from its set of vertices and deleting all edges emerging from v from the set of edges. Also, $N_G(v)$ will denote the neighborhood of v in G, that is, the set of all $u \in V(G)$ such that $\{u, v\} \in E(G)$. If v and w are two distinct vertices of G then vw will denote the two-element set $\{v, w\}$, $G - vw$ will denote the graph $(V(G), E(G) \setminus \{vw\})$, and $G + vw$ will denote the graph $(V(G), E(G) \cup \{vw\})$. Note that (by definition) if $xy \in E(G)$ then $G + xy = G$ and if $xy \notin E(G)$ then $G - xy = G$. For any graph G, $\overline{G}$ will denote the complement graph $(V(G), \binom{V(G)}{2} \setminus E(G))$. Also, for $U \subseteq V(G)$, G_U will denote the subgraph of G induced on U, so $G_U = (U, E_U)$, where E_U is the set of all edges $vw \in E(G)$ such that $\{v, w\} \subseteq U$.

The *face lattice* of a simplicial complex is the poset of simplices ordered by inclusion. In this article we will include the empty face as $\hat{0}$, but not an artifical top element $\hat{1}$. All homology groups discussed in this paper have integer coefficients.

3. Forman's discrete Morse theory

In this section we give a combinatorial description of Forman's discrete Morse theory for simplicial complexes (see [Fo1]). This description is originally due to Chari (see [Ch1]).

For a poset P, we define $D(P) = (P, A(P))$ to be the directed graph obtained by directing each edge in the Hasse diagram of P downwards, so there is an arc (x, y) in $A(P)$ if and only if x covers y in P. For $M \subseteq A(P)$, we define $D_M(P)$ to be the directed graph obtained from $D(P)$ by reversing the direction of all the arcs in M, so $D_M(P) = (P, A_M(P))$, where

$$A_M(P) = (A(P) \setminus M) \cup \{(x,y) : (y,x) \in M\}.$$

If P is the face lattice (including the empty face) of a simplcial complex Σ, we will write $D(\Sigma)$ for $D(P)$ and $D_M(\Sigma)$ for $D_M(P)$. We call $M \subseteq A(P)$ an *acyclic matching* if M satisfies the conditions

(M1) M is a matching, that is, each element of P is an endpoint of at most one arc in M, and

(M2) $D_M(P)$ has no directed cycles.

If $M \subseteq A(P)$ is an acyclic matching, the *critical points* of M are those $x \in P$ such that there is no arc in M having x as an endpoint. M is an *acyclic perfect matching* if M is acyclic and has no critical points. A *Morse matching* on a simplicial complex Σ is an acyclic matching in $D(\Sigma)$.

The next theorem is a special case of the main result of [Fo1].

Theorem 3.1 ([Fo1, Corollary 3.5]). *Let M be a Morse matching on a nonempty simplicial complex Σ. Assume that $\emptyset$ is not a critical point of M. Then Σ is homotopy equivalent to a CW-complex which has one 0-cell and, for $k \geq 0$, one additional k-cell for each k-dimensional face of Σ which is a critical point of M.*

It follows immediately that if M is a Morse matching on Σ and there is some k such that each critical point of M is k-dimensional then Σ is homotopy equivalent to a wedge of k-dimensional spheres.

The next three results are useful in verifying that a given M is a Morse matching, and will be used repeatedly in what follows. The first result, which allows one to produce a Morse matching on a complex Σ by piecing together acyclic matchings on subposets of the face lattice of Σ, first appeared in [Jo]. The second result first appeared in [Sh1]. Both have straightforward proofs. The third result gives a general method for defining Morse matchings. It gives a uniform explanation of the ad hoc constructions of Morse matchings found in [BBLSW], [Jo] and [Sh1, Sh2].

Lemma 3.2 (**Cluster Lemma** - [Jo, Lemma 2.2]). *Let $P_1, \dots, P_r$ be pairwise disjoint, order convex subposets of P. For each $i \in [r]$, let M_i be an acyclic matching on $D(P_i)$. Define a relation on the P_i by $P_i \leq_c P_j$ if there exist $x \in P_i$ and $y \in P_j$ such that $x \leq y$. Assume that the P_i satisfy the condition*

($\mathcal{P}$) *The relation $\leq_c$ defines a partial order on the P_i's.*

Then

$$M := \bigcup_{i=1}^{r} M_i$$

is an acyclic matching on $D(P)$.

Lemma 3.3 (**Cycle Lemma** - [Sh1, Proposition 3.1]). *Let P be an order convex subposet of the face lattice of a simplicial complex Σ and assume that $M \subseteq A(P)$ satisfies condition (M1). Then every directed cycle in $D_M(P)$ is of the form $\sigma_1, \tau_1, \sigma_2, \tau_2, \dots, \sigma_{r-1}, \tau_{r-1}, \sigma_r = \sigma_1$, where*

1. *$r \geq 3$,*
2. *for each $i \in [r-1]$, there is some $x_i \in \tau_i$ such that $\tau_i = \sigma_i \cup \{x_i\}$ and $(\tau_i, \sigma_i) \in M$, and*

3. *for each* $i \in [r-1]$, *there is some* $y_i \in \tau_i$ *such that* $\sigma_{i+1} = \tau_i \setminus \{y_i\}$,
4. *the multisets* $\{x_i : i \in [r]\}$ *and* $\{y_i : i \in [r]\}$ *are equal.*

Lemma 3.4 (Pairing Lemma). *Let* Σ *be a simplicial complex on a partially ordered vertex set* $(V, \preceq)$. *Let* P *be an order convex subposet of the face lattice of* Σ. *For a function* $f : P \to V$, *set*

$$P_f := \{\sigma \in P : f(\sigma) \notin \sigma\}.$$

For $\sigma \in P_f$, *set*

$$\sigma^+ := \sigma \cup \{f(\sigma)\},$$

and for $\tau \in P \setminus P_f$ *set*

$$\tau^- = \tau \setminus \{f(\tau)\}.$$

Assume that f *satisfies the conditions*
(A) *if* $\sigma \in P_f$ *then* $\sigma^+ \in P$,
(B) *if* $\sigma \in P_f$ *then* $f(\sigma^+) = f(\sigma)$,
(C) *if* $\tau \in P \setminus P_f$ *and* $\tau^- \in P$ *then* $f(\tau^-) = f(\tau)$, *and*
(D) *if* $x \in \sigma \in P_f$ *and* $\sigma^+ \setminus \{x\} \in P$ *then* $f(\sigma) \preceq f(\sigma^+ \setminus \{x\})$.
Then

$$M_f := \left\{(\sigma^+, \sigma) : \sigma \in P_f\right\}$$

is an acyclic matching on $D(P)$, *and the critical points of* M_f *are those* $\rho \in P \setminus P_f$ *such that* $\rho \setminus \{f(\rho)\} \notin P$.

Proof. Since f satisfies condition (A), the endpoints of each arc in M_f lie in P. For any arc (ρ, τ) in M_f we have $\rho \in P \setminus P_f$ and $\tau \in P_f$. It follows that for $\sigma \in P_f$ there is exactly one arc in M_f with σ as an endpoint, namely, (σ^+, σ). Say $\tau \in P \setminus P_f$. If $\tau^- \in P$ then, since f satisfies condition (C), we have $\tau^- \in P_f$ and $(\tau, \tau^-) \in M_f$. Since $\tau \in P \setminus P_f$, there is no arc $(\rho, \tau) \in M$, and any arc $(\tau, \sigma) \in M_f$ satisfies $\sigma \in P_f$ and $\tau = \sigma \cup \{f(\sigma)\}$. Since f satisfies condition (B), we have $\sigma = \tau^-$. Therefore, if $\tau^- \in P$ then (τ, τ^-) is the unique arc in M_f having τ as an endpoint, while if $\tau^- \notin P$ there is no arc in M having τ as an endpoint. Thus M_f satisfies condition (M1) and the elements of P which are not covered by an arc in M_f are the elements which are claimed to be critical points of M_f. It remains to show that M_f satisfies condition (M2). Assume for contradiction that

$$\sigma_1, \tau_1, \dots, \sigma_{r-1}, \tau_{r-1}, \sigma_r = \sigma_1$$

is a cycle in $D_{M_f}(P)$, which satisfies the conditions of Cycle Lemma 3.3. Then for each $i \in [r-1]$ we have $(\tau_i = \sigma_i^+, \sigma_i) \in M_f$ and $\sigma_{i+1} = \tau_i \setminus y_i$, where $y_i \neq x_i = f(\sigma_i)$. Since f satisfies condition (D), we have

$$f(\sigma_1) = x_1 \preceq f(\sigma_2) = x_2 \preceq \ldots \preceq f(\sigma_r) = x_r = x_1,$$

so $x_i = x_1$ for all $i \in [r-1]$. Now, since condition 4 of Cycle Lemma 3.3 is satisfied, we have $y_i = x_1$ for all i, contradicting our assumption that $y_i \neq x_i$. $\square$

Convention: When applying Lemma 3.4 to a complex Σ of graphs on vertex set $[n]$, so that the vertices of Σ are the elements of $\binom{[n]}{2}$, we will assume unless otherwise stated that the order $\preceq$ on $\binom{[n]}{2}$ is the lexicographic order, so $\{a < b\} \preceq \{c < d\}$ if either $a < c$ or $a = c$ and $b \leq d$.

4. THE COMPLEX OF BIPARTITE GRAPHS

In this section we will prove Theorem 1.1 by producing a Morse matching on $D(\Gamma_n^2)$ whose critical points are the trees in the set $\mathcal{T}(n)$ defined in Section 1. As we will discuss in more detail below, a more general result has been obtained by Chari (see [Ch2]), who produced a Morse matching whose critical points are trees for the complex of bipartite subgraphs of any graph G such that each edge of G is contained in an odd cycle.

For the reader's convenience, we repeat the definition of $\mathcal{T}(n)$, recalling that $d(x, y)$ is the length of the shortest path between two vertices x, y of G (with $d(x, y) = \infty$ if no such path exists). For any graph $G = ([n], E)$ and $i \in \mathbb{N} \cup \{\infty\}$, we define

$$D_i(G) := \{x \in [n] \setminus \{1\} : d(1, x) = i\},$$

and

$$md(G) = max \{i \in \mathbb{N} : D_i(G) \neq \emptyset\}.$$

For $1 \leq i \leq md(G)$, define

$$a_i(G) := minD_i(G).$$

Now $\mathcal{T}(n)$ is defined to be the set of all trees on vertex set $[n]$ satisfying

1. for $1 \leq i < md(T)$, the only element of $D_i(T)$ which has neighbors in $D_{i+1}(T)$ is $a_i(T)$,
2. for $1 \leq i < md(T)$, there is some element of $\bigcup_{j \geq i} D_j(T)$ which is larger than $a_i(T)$, and
3. $|D_{md(T)}(T)| > 1$.

As stated above, we will prove the following result, from which Theorem 1.1 follows immediately upon application of Theorem 3.1.

Theorem 4.1. *For $n \geq 2$, there is a Morse matching on $D(\Gamma_n^2)$ whose critical points are all trees $T \in \mathcal{T}(n)$.*

Before proving Theorem 4.1 we will discuss the general result of Chari mentioned above. Fix a graph $G = ([n], E(G))$ and a linear order $\preceq$ on $E(G)$. Let T be a spanning tree of G. For an edge $e \in E(T)$, let $A(e), B(e)$ be the connected components of $T - e$. Then e is called internally active (with respect to T) if e is least (with respect to $\preceq$) among all edges of G which have one endpoint in $A(e)$ and one endpoint in $B(e)$. The internal activity $IA(T)$ is the set of all $e \in e(T)$ which are internally active with respect to T. Let $e \in E(G) \setminus E(T)$ be an edge such that the unique cycle $C(e)$ in $T + e$ has odd length. Then e is called oddly externally active (with respect to T) if e is the least element of $C(e)$. The odd external activity $OEA(T)$ is the set of all $e \in E(G) \setminus E(T)$ which are oddly externally active with respect to T. Using a result of C. Colbourn and W. Pulleyblank (see [CP]), Chari proves the following theorem.

Theorem 4.2 ([Ch2])**.** *For any graph G such that each $e \in E(G)$ is contained in an odd cycle in G, let $\Gamma^2(G)$ be the complex of bipartite subgraphs $H = (V(G), E(H))$ of G. Then $D(\Gamma^2(G))$ admits a Morse matching whose critical points are those spanning trees T of G such that $IA(T) = OEA(T) = \emptyset$.*

If some $e \in E(G)$ is not contained in any odd cycle then

$$\{(H, H - e) : e \in E(H)\}$$

is a Morse matching on $\Gamma^2(G)$ with no critical points, so $\Gamma^2(G)$ is contractible (actually, collapsible). Note that the set of critical points of the Morse matching given in Theorem 4.2 depends on the chosen linear order $\preceq$ on $E(G)$. It is straightforward to show that when $n = 4$ there exists no linear ordering of the edges of K_4 such that the trees in $\mathcal{T}(4)$ are the trees satisfying $IA(T) = OEA(T) = \emptyset$. Indeed, if for each $i \in [4]$ we let T_i be the unique tree on vertex set $[4]$ such that $|N_G(i)| = 3$ then $\mathcal{T}(4) = \{T_1, T_2, T_3\}$. If $IA(T_1) = OEA(T_1) = \emptyset$ then the given linear order on $E(K_4)$ must be one of

$$34 \prec 13 \prec 23 \prec 12 \prec 24 \prec 14$$

or

$$12 \prec 23 \prec 13 \prec 34 \prec 14 \prec 24.$$

If the first order is chosen then 13 is externally active with respect to T_2, while if the second order is chosen then 12 is internally active with respect to T_2. Therefore we cannot have $IA(T_i) = OEA(T_i)$ for $i = 1$ and $i = 2$ simultaneously. Enumerating the trees with no internal activity and no odd external activity in an arbitrary graph G seems quite difficult, and it would be interesting if one could find another Morse matching on $D(\Gamma^2(G))$ (for arbitrary G) whose critical points are easy to enumerate.

We now prove Theorem 4.1, by induction on n. If $n = 2$ then Γ^2_n consists of the empty graph and K_2, so there is a Morse matching on $D(\Gamma^2_2)$ with no critical points. Since no tree on two vertices satisfies condition 3 in the definition of $\mathcal{T}(n)$, we see that $\mathcal{T}(2) = \emptyset$ and the theorem holds. Now assume $n > 2$. Define

- $I_n := \{G \in \Gamma^2_n : N_G(1) = \emptyset\}$,
- $J_n := \{G \in \Gamma^2_n \setminus I_n :$ if $x \in D_2(G)$ then $N_G(x) \cap D_1(G) = \{a_1(G)\}\}$, and
- for $2 \leq i \leq n$, $J_n(i) := \{G \in J_n : a_1(G) = i\}$.

So, $G \in I_n$ if and only if 1 is isloated in G and $G \in J_n$ if and only if every path $1, x, y$ in G satisfies $x = a_1(G)$. Also, $J_n = \bigcup_{i=2}^n J_n(i)$, and $G \in J_n(n)$ if and only if $N_G(1) = \{n\}$. Note that I_n is a subcomplex of Γ^2_n, as is $I_n \cup \bigcup_{i \in S} J_n(i)$ for every $S \subseteq \{2, \ldots, n\}$. Also, the pairwise disjoint, order convex subsets I_n, $J_n(2), \ldots, J_n(n)$, $\Gamma^2_n \setminus (I_n \cup J_n)$ cover Γ^2_n and satisfy condition $\mathcal{P}$ of Cluster Lemma 3.2.

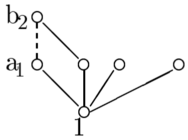
Set $P = P_1 := I_n \cup J_n(n)$ and define $f = f_1 : P \to \binom{[n]}{2}$ by $f_1(G) = \{1, n\}$ for all G. Then f satisfies conditions (A)-(D) of Pairing Lemma 3.4, with $P_f = I_n$. Since $G - \{1, n\} \in I_n$ for each $G \in J_n(n)$, $M_1 := M_f$ is an acyclic perfect matching on P_1.

Now set $P = P_n := \Gamma^2_n \setminus (I_n \cup J_n)$. Note that $G \in P_n$ if and only if the set

$$R_2(G) := \{x \in D_2(G) : N_G(x) \cap D_1(G) \neq \{a_1(G)\}\}$$

is not empty. For $G \in P_n$, set $b_2(G) = minR_2(G)$. Note that in any proper 2-coloring χ of G

$$\chi(a_1(G)) \neq \chi(1) = \chi(b_2(G)).$$



Therefore, if $G \in P_n$ with $\{a_1(G), b_2(G)\} \notin E(G)$ then $G + \{a_1(G), b_2(G)\} \in P_n$. It follows that if we define $f = f_n : P_n \to \binom{[n]}{2}$ by

$$f_n(G) := \{a_1(G), b_2(G)\},$$

then f satisfies condition (A) of Pairing Lemma 3.4. Adding or removing $f(G)$ from $E(G)$ changes neither $D_1(G)$ nor $R_2(G)$, so f satisfies conditions (B) and (C) and every element of P is an endpoint of some arc in M_f. As stated in Section 3, we order the vertices of Γ_n^2, which are the edges of K_n, lexicographically. Say $G \in P_f$ and $H = G^+ - e$ for some $e \in E(G) \setminus \{f(G)\}$. If $e \neq \{1, a_1(G)\}$ then $a_1(H) = a_1(G)$. Also, $R_2(H) \subseteq R_2(G)$, so either $R_2(H) = \emptyset$ and $H \notin P$ or $b_2(H) \geq b_2(G)$. Say $e = \{1, a_1(G)\}$. If $H \in P$ then we again have $R_2(H) \subseteq R_2(G)$, since the edge $f(G)$ is not in any path $1, x, y$ in H. Thus $b_2(H) \geq b_2(G)$ and $a_1(H) > a_1(G)$. Therefore, f satisfies condition (D) and $M_n := M_f$ is an acyclic perfect matching on P_n.

We use our inductive hypothesis to define our matching on the remaining parts of Γ_n^2. Fix $i \in \{2, \ldots, n-1\}$ and set $P = P_i := J_n(i)$. For $X \subseteq [n] \setminus [i]$, set

$$P(X) := \{G \in P : N_G(1) = X \cup \{i\}\}.$$

The order convex subsets $P(X)$ of P satisfy condition $\mathcal{P}$ of Cluster Lemma 3.2, and we will define an acyclic matching M_i on $D(P_i)$ by defining an acyclic matching $M(X)$ on each $D(P(X))$ and then setting $M_i := \bigcup_X M(X)$. Set $Y = [n] \setminus (X \cup \{1\})$ and let $m = |Y|$. Note that $G = ([n], E) \in P(X)$ if and only if G satisfies the conditions

- $N_G(1) = X \cup \{i\}$,
- $N_G(x) = \{1\}$ for all $x \in X$, and
- G_Y is bipartite.

If $m = 1$ then $i = 2$ and $N_G(1) = [n] \setminus \{1\}$ for all $G \in P(X)$. It follows that the unique element of $P(X)$ is the tree T in which vertex 1 is the only vertex which is not a leaf. In this case we define $M(X) = \emptyset$, so T is a critical point of $M(X)$.

Now assume $m > 1$. Define $\phi : Y \to [m]$ by

- $\phi(i) = 1$, and
- if $x, y \in Y \setminus \{i\}$ with $x < y$ then $1 < \phi(x) < \phi(y)$.

Let Γ_Y^2 be the complex of bipartite graphs on vertex set Y. Then ϕ determines a simplicial isomorphism from Γ_Y^2 to Γ_m^2. For $G \in P(X)$ let $\pi(G) = G_Y$. By the characterization of $P(X)$ given just above, $\phi\pi$ determines a digraph isomorphism between $D(P(X))$ and $D(\Gamma_m^2)$. In this case, by the inductive hypothesis there exists a Morse matching M^m on $D(\Gamma_m^2)$ whose critical points are the elements of $\mathcal{T}(m)$. Set $\tau = (\phi\pi)^{-1}$ and define

$$M(X) := \{(\tau(G), \tau(H)) : (G, H) \in M^m\}.$$

Then $M(X)$ is an acyclic matching on $P(X)$. The critical points of $M(X)$ are those trees T satisfying the conditions

- $a_1(T) = i$,
- $N_T(1) = X \cup \{i\}$,
- $N_T(x) = \{1\}$ for all $x \in X$,
- for $2 \leq r \leq md(T)$, there is some element of $\cup_{j>r} D_j(T)$ which is larger than $a_r(T)$, and
- $|D_{md(T)}(T)| > 1$.

In other words, the critical points of $M(X)$ are all $T \in \mathcal{T}(n)$ such that $a_1(T) = i$ and $D_1(T) = X \cup \{i\}$. Now the critical points of $M_i = \bigcup_X M(X)$ are exactly those trees $T \in \mathcal{T}_n$ such that $a_1(T) = i$. Finally, by Cluster Lemma 3.2,

$$M^n := \bigcup_{i=1}^{n} M_i$$

is a Morse matching on $D(\Gamma_n^2)$ whose critical points are the elements of $\mathcal{T}(n)$ and the proof of Theorem 4.1 is complete.

5. Complexes of highly colorable graphs

In this section, we will prove Theorems 1.3 and 1.4 by finding Morse matchings with the appropriate number of critical points in the appropriate dimensions. First we will find a Morse matching on Γ_n^t for arbitrary n, t whose critical points are graphs which are quite large, given that they are t-colorable, thereby proving Proposition 1.5. The notation introduced in the next definition will be used repeatedly.

Definition: For a graph $G = ([n], E)$, G' is the subgraph of G induced on $[n] \setminus \{1\}$.

For $n \in \mathbb{N}$ and $t \in [n]$, define

$$\mathcal{I}(n,t) := \left\{G \in \Gamma_n^t : \chi(G') \le t-1\right\}.$$

Note that if H is any graph on vertex set $W := \{2, 3, \dots, n\}$ with $\chi(H) \le t-1$ and G is any graph on vertex set $[n]$ such that the subgraph of G induced on W is H then $G \in \mathcal{I}(n,t)$.

For $G \in \Gamma_n^t \setminus \mathcal{I}(n,t)$, define

$$\Theta(G) := \{v \in V(G') : |N_{G'}(v)| < t-1\},$$

and

$$\mathcal{J}(n,t) := \left\{G \in \Gamma_n^t \setminus \mathcal{I}(n,t) : \Theta(G) \ne \emptyset\right\}.$$

Also, define

$$\mathcal{K}(n,t) := \mathcal{I}(n,t) \bigcup \mathcal{J}(n,t).$$

Lemma 5.1. *Let $n \in \mathbb{N}$ and $t \in [n]$. Then $\mathcal{K}(n,t)$ is a subcomplex of Γ_n^t and $D(\mathcal{K}(n,t))$ admits an acyclic perfect matching.*

Proof. Removing an edge from any graph $G = ([n], E)$ cannot increase $\chi(G')$ or increase $|N_{G'}(v)|$ for $v \in V(G')$, so both $\mathcal{I}(n,t)$ and $\mathcal{K}(n,t)$ are subcomplexes of Γ_n^t. By Cluster Lemma 3.2, it is now sufficient to produce acyclic perfect matchings on $\mathcal{I}(n,t)$ and $\mathcal{J}(n,t)$.

We begin with $\mathcal{I}(n,t)$. Define

$$\mathcal{I}_0(n,t) := \{G \in \mathcal{I}(n,t) : \{1,2\} \notin E(G)\}.$$

For $G \in \mathcal{I}_0(n,t)$, set

$$G^+ := G + \{1,2\}.$$

Then $G' = (G^+)'$, so $G^+ \in \mathcal{I}(n,t)$. For $G \in \mathcal{I}(n,t) \setminus \mathcal{I}_0(n,t)$, set

$$G^- := G - \{1,2\}.$$

Then $G^- \in \mathcal{I}_0(n,t)$ and $G = (G^-)^+$, so

$$\mathcal{I}(n,t) = \mathcal{I}_0(n,t) \bigcup \left\{G^+ : G \in \mathcal{I}_0(n,t)\right\}.$$

Now define $f(G) := \{1,2\}$ for all $G \in \mathcal{I}(n,t)$. Setting $P = \mathcal{I}(n,t)$ and using the language of Lemma 3.4, we have $P_f = \mathcal{I}_0(n,t)$. By the arguments given just above, f satisfies conditions (A),(B) and (C) of Pairing Lemma 3.4. Since f is a constant function, f satisfies condition (D). It now follows that

$$M_f = \left\{(G, G^-) : G \in \mathcal{I}(n,t) \setminus \mathcal{I}_0(n,t)\right\}$$

is an acyclic perfect matching on $D(\mathcal{I}(n,t))$.

For $G \in \mathcal{J}(n,t)$, define

$$\theta(G) := \max \Theta(G)$$

and

$$\mathcal{J}_0(n,t) := \{G \in \mathcal{J}(n,t) : \{1, \theta(G)\} \notin E(G)\}.$$

For $G \in \mathcal{J}_0(n,t)$, set

$$G^+ := G + \{1, \theta(G)\}.$$

First note that $|N_{G^+}(\theta(G))| < t$, so one can obtain a proper t-coloring of G^+ from a proper t-coloring χ of G by assigning to $\theta(G)$ any color i such that $\chi^{-1}(i) \cap N_{G^+}(\theta(G)) = \emptyset$. Hence $G^+ \in \Gamma_n^t$. Furthermore $G' = (G^+)'$, so $G^+ \in \mathcal{J}(n,t)$ and $\theta(G^+) = \theta(G)$. For $G \in \mathcal{J}(n,t) \setminus \mathcal{J}_0(n,t)$, set

$$G^- = G - \{1, \theta(G)\}.$$

Then $G^- \in \mathcal{J}(n,t)$ and $\theta(G^-) = \theta(G)$, so $G^- \in \mathcal{J}_0(n,t)$. It follows that

$$\mathcal{J}(n,t) = \mathcal{J}_0(n,t) \bigcup \left\{G^+ : G \in \mathcal{J}_0(n,t)\right\}.$$

For $G \in \mathcal{J}(n,t)$, define $f(G) := \{1, \theta(G)\}$. Then, again using the language of Pairing Lemma 3.4 and setting $P = \mathcal{J}(n,t)$, we have $P_f = \mathcal{J}_0(n,t)$. By the arguments given just above, f satisfies conditions (A),(B) and (C) of the Lemma. Say $G \in \mathcal{J}_0(n,t)$ and $e \in E(G) \setminus \{\{1, \theta(G)\}\}$ with $G^+ - e \in \mathcal{J}(n,t)$. For any $H \in \mathcal{J}(n,t)$ and $d \in E(H)$ such that $H - d \in \mathcal{J}(n,t)$, we have $\theta(H) \le \theta(H - d)$. Thus

$$f(G) = f(G^+) \preceq f(G^+ - e),$$

so f satisfies condition (D). Therefore,

$$M_f = \left\{(G, G^-) : G \in \mathcal{J}(n,t) \setminus \mathcal{J}_0(n,t)\right\}$$

is an acyclic perfect matching on $D(\mathcal{J}(n,t))$. $\square$

Note that we now have a Morse matching on Γ_n^t whose critical points are the elements of $\Gamma_n^t \setminus \mathcal{K}(n,t)$, that is, the graphs $G \in \Gamma_n^t$ which satisfy the conditions

- $\chi(G') = t$, and
- $|N_{G'}(v)| \ge t - 1$ for all $v \in V(G')$.

Each such graph has at least $\lceil \frac{(n-1)(t-1)}{2} \rceil$ edges, and Proposition 1.5 now follows from Theorem 3.1. When $t = n - k$ and k is small, the graphs satisfying these conditions have highly restricted structure, as we shall see. Recall the notation $\overline{G}$ for the complement of G, that is the graph on the same vertex set with complementary edge set.

Say $k = 2$, so $t = n - 2$. If $\overline{G'}$ contains two nonintersecting edges vw and xy then we can construct a proper $t-1$-coloring of G' by assigning one color to v and w, another color to x and y and $t - 3 = n - 5$ additional colors to the remaining $n - 5$ vertices of G'. Therefore, if $G \in \Gamma_n^t \setminus \mathcal{K}(n,t)$ then $\overline{G'}$ does not contain two nonintersecting edges. Also, each vertex of G' has degree at least $n - 3$ in G', so $\overline{G'}$ has no vertex of degree more than one. On the other hand, $\overline{G'}$ must contain at least one edge, since otherwise $\chi(G') = n - 1$. Therefore,

$$\Gamma_n^t \setminus \mathcal{K}(n,t) = \left\{G \in \Gamma_n^t : |E(\overline{G'})| = 1\right\}.$$

For $\{x,y\} \in \binom{V(G')}{2}$, define

$$P(x,y) := \left\{G \in \Gamma_n^t : E(\overline{G'}) = \{xy\}\right\}.$$

The pairwise disjoint, order convex subsets $\mathcal{K}(n,t)$ and $P(x,y)$ (where $\{x,y\}$ runs through all of $\binom{V(G')}{2}$)) cover Γ_n^t and satisfy condition $\mathcal{P}$ of Cluster Lemma 3.2, so we will concentrate on the posets $P(x,y)$ and then apply the lemma.

Fix $\{x,y\} \in \binom{V(G')}{2}$ with $x < y$. If $G \in P(x,y)$ then every proper t-coloring of G' assigns one color to both of x,y and all of the $n-3$ remaining colors to the remaining $n-3$ vertices of G'. It follows that

- $N_G(1) \subseteq V(G') \setminus \{x,y\}$, or
- there is some $z \in V(G') \setminus \{x,y\}$ such that $z \notin N_G(1)$.

Conversely, if $G = ([n], E)$ with $E(\overline{G'}) = \{xy\}$ and $N_G(1)$ satisfies either of the two conditions itemized just above then $G \in P(x,y)$. Let $G(x,y)$ be the graph in $P(x,y)$ defined by

$$N_{G(x,y)}(1) := V(G') \setminus \{x,y\}.$$

Set

$$P_1(x,y) := P(x,y) \setminus \{G(x,y)\},$$

and

$$P_0(x,y) := \{G \in P_1(x,y) : 1x \notin E(G)\}.$$

Then $P_1(x,y)$ is an order ideal in $P(x,y)$, and it is straightforward to show that

$$M(x,y) := \{(G + 1x, G) : G \in P_0(x,y)\}$$

is an acyclic perfect matching on $D(P_1(x,y))$. Thus $M(x,y)$ is an acyclic matching on $D(P(x,y))$ whose unique critical point is $G(x,y)$. Now applying Cluster Lemma 3.2 gives the following result.

Theorem 5.2. *There is a Morse matching on Γ_n^{n-2} whose critical points are the graphs $G(x,y)$ defined by*

$$E(\overline{G(x,y)}) := \{1x, 1y, xy\},$$

for all $\{x,y\} \in \binom{[n-1]}{2}$. Therefore, Γ_n^{n-2} has the homotopy type of a wedge of $\binom{n-1}{2}$ spheres of dimension $\binom{n}{2} - 4$.

To handle the case $t = n - 3$ we introduce additional notation. Let $j \in \mathbb{N}$. Then Ma_j will denote a graph with $2j$ vertices and j pairwise nonintersecting edges. If these edges are $x_1y_1, \ldots, x_jy_j$ then the graph may be denoted by $\mathsf{Ma}(x_1, y_1; \ldots; x_j, y_j)$. Also, Cy_j will denote a graph with j vertices and j edges which form a cycle of length j. If the edges are $x_1x_2, \ldots, x_{j-1}x_j, x_jx_1$ then the graph may be denoted by $\mathsf{Cy}(x_1, \ldots, x_j)$. Finally, Pa_j will denote a graph with $j+1$ vertices and j edges which form a path of length j. If these edges are $x_1x_2, \ldots, x_jx_{j+1}$ then the graph may be denoted by $\mathsf{Pa}(x_1, \ldots, x_{j+1})$.

If G, H are graphs with disjoint vertex sets then $G + H$ will denote the graph $(V(G) \cup V(H), E(G) \cup E(H))$. For any graph G, set

$$U(G) := \{v \in V(G) : N_G(v) \neq V(G) \setminus \{v\}\}\,.$$

Note that if $G \neq K_n$ then $U(G) \neq \emptyset$. A special case of the following lemma was used when we examined the case $t = n - 2$.

Lemma 5.3. *Let $n \in \mathbb{N}$ and let $t \in [n]$. Define*

$$\mathcal{L}(n,t) := \left\{G \in \Gamma_n^t \setminus \mathcal{K}(n,t) : V(G') \setminus U(G') \not\subseteq N_G(1)\right\}.$$

For $G \in \mathcal{L}(n,t)$, define

$$\alpha(G) := \max U(G').$$

Set

$$\mathcal{L}_0(n,t) := \{G \in \mathcal{L}(n,t) : \{1, \alpha(G)\} \notin E(G)\}\,.$$

Then

$$L(n,t) := \{(G + \{1, \alpha(G)\}, G) : G \in \mathcal{L}_0(n,t)\}$$

is an acyclic perfect matching on $D(\mathcal{L}(n,t))$.

Proof. We will apply Pairing Lemma 3.4 to the order ideal $P := \mathcal{L}(n,t)$ of $\Gamma_n^t \setminus \mathcal{K}(n,t)$. Define $f : \mathcal{L}(n,t) \to \binom{[n]}{2}$ by $G \mapsto \{1, \alpha(G)\}$. Then $P_f = \mathcal{L}_0(n,t)$. Let $G \in P_f$ and let χ be a proper t-coloring of G'. We can extend χ to a proper t-coloring of $G^+ := G + \{1, \alpha(G)\}$ by giving vertex 1 the same color as any element of $(V(G') \setminus U(G')) \setminus N_G(1)$. Therefore, $G^+ \in \Gamma_n^t$. Since changing $N_G(1)$ does not change G', we have $G^+ \in \mathcal{L}(n,t)$. Moreover, $\alpha(G^+) = \alpha(G)$. It follows that f satisfies conditions (A),(B) and (C) of Pairing Lemma 3.4. Finally, if $G \in P_f$ and $e \in E(G)$ with $H := G^+ - e \in \mathcal{L}(n,t)$ then $\alpha(H) \geq \alpha(G^+)$, so

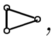
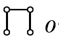
$$f(G) = f(G^+) \preceq f(H).$$

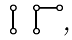
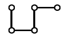
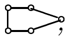
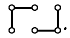
Therefore, f satisfies condition (D). $\square$

Now $\mathcal{L}(n,t)$ is an order ideal in $\Gamma_n^t \setminus \mathcal{K}(n,t)$, and we now have a Morse matching on $D(\Gamma_n^t)$ whose critical points are the graphs in $\Gamma_n^t \setminus (\mathcal{K}(n,t) \cup \mathcal{L}(n,t))$. These critical graphs form an order filter in Γ_n^t, so if we produce an acyclic matching on the digraph of this filter we will have a Morse matching on Γ_n^t, by Cluster Lemma 3.2.

Now assume $t = n - 3$. If $G \notin \mathcal{K}(n,t)$ then every vertex of G' has degree at least $n - 4$, so every vertex of $\overline{G'}$ has degree $0, 1$ or 2. It follows that the induced subgraph $\overline{G'}_{U(G')}$ is a union of paths and cycles. Let $G \in \Gamma_n^t$. If $\overline{G'}$ contains a subgraph isomorphic to either of Ma_3 or $\mathsf{Cy}_3 + \mathsf{Ma}_1$ then $\chi(G') \leq n - 4$ and $G \in \mathcal{K}(n,t)$. Conversely, in order that G' be $(n-3)$-colorable it is necessary that $\overline{G'}$ contain either Ma_2 or Cy_3. The next proposition follows.

Proposition 5.4. *Let $n \in \mathbb{N}$ with $n \geq 4$. Set $t = n - 3$ and let $G \in \Gamma_n^t$. Then $G \in \Gamma_n^t \setminus \mathcal{K}(n,t)$ if and only if one of the following conditions hold.*

- $|U(G')| = 3$ *and* $\overline{G'}_{U(G')}$ *is isomorphic to* Cy_3 ,
- $|U(G')| = 4$ *and* $\overline{G'}_{U(G')}$ *is isomorphic to one of* Ma_2 , Pa_3  *or* Cy_4 ,

- $|U(G')| = 5$ *and* $\overline{G'}_{U(G')}$ *is isomorphic to one of* $\mathsf{Ma}_1 + \mathsf{Pa}_2$ , Pa_4  *or* Cy_5 ; *or*
- $|U(G')| = 6$ *and* $\overline{G'}_{U(G')}$ *is isomorphic to* $\mathsf{Pa}_2 + \mathsf{Pa}_2$ .

For each graph H such that $V(H) \subseteq [2,n]$ and $|V(H)| \in \{3,4,5,6\}$, set

$$R(H) := \left\{G \in \Gamma_n^t \setminus (\mathcal{K}(n,t) \cup \mathcal{L}(n,t)) : \overline{G'}_{U(G')} = H\right\}.$$

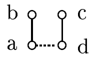
For $\{a < b < c < d\} \subseteq [2,n]$, set

$$Q(a,b,c,d) := \left\{G \in \Gamma_n^t \setminus (\mathcal{K}(n,t) \cup \mathcal{L}(n,t)) : U(G') = \{a,b,c,d\}\right\} = \bigcup_{V(H)=\{a,b,c,d\}} R(H).$$

The pairwise disjoint, order convex subsets $R(H)$ and $Q(a,b,c,d)$ (where H runs over all graphs of the isomorphism types described in proposition 5.4 such that $|V(H)| \in \{3,5,6\}$ and $\{a,b,c,d\}$ runs over $\binom{[2,n]}{4}$) cover $\Gamma_n^t \setminus (\mathcal{K}(n,t) \cup \mathcal{L}(n,t))$ and satisfy condition $(\mathcal{P})$ of the Cluster Lemma. We will find useful Morse matchings on each of these subsets and apply the lemma.

We begin with the posets $Q(a,b,c,d)$. Fix $a < b < c < d$. Set

$$\begin{aligned} Q_1 &:= R(\mathsf{Ma}(a,b;c,d)) \bigcup R(\mathsf{Pa}(b,a,d,c)), \\ Q_2 &:= R(\mathsf{Ma}(a,c;b,d)) \bigcup R(\mathsf{Pa}(c,a,d,b)), \textit{and} \\ Q_3 &:= R(\mathsf{Ma}(a,d;b,c)) \bigcup R(\mathsf{Pa}(d,a,c,b)). \end{aligned}$$



Then the Q_i are pairwise disjoint order filters in $Q(a,b,c,d)$. We will find an acyclic matching on $D(Q(a,b,c,d))$ whose critical points are those graphs not contained in $\bigcup_{i=1}^3 Q_i$ by finding an acyclic perfect matching on each $D(Q_i)$. The $D(Q_i)$ are pairwise isomorphic, so it suffices to find an acyclic perfect matching on $D(Q_1)$. If $G \in R(\mathsf{Ma}(a,b;c,d))$ then every proper $(n-3)$-coloring of G' is obtained by assigning one color to vertices a,b, another color to vertices c,d and all of the remaining $n-5$ colors to the remaining $n-5$ vertices of G'. Such a coloring can be extended to a proper coloring of G, so $N_G(1) \cap \{a,b,c,d\}$ must be contained in either $\{a,b\}$ or $\{c,d\}$. Conversely, if $G = ([n], E)$ satisfies

1. $E(\overline{G'}) = \{ab, cd\}$,
2. $V(G') \setminus \{a,b,c,d\} \subseteq N_G(1)$ and
3. $N_G(1) \cap \{a,b,c,d\}$ is contained in one of $\{a,b\}$ or $\{c,d\}$

then $G \in R(\mathsf{Ma}(a,b;c,d))$, so the three conditions just given characterize $R(\mathsf{Ma}(a,b;c,d))$. Now a similar argument shows that $R(\mathsf{Pa}(b,a,d,c))$ is characterized by conditions 2,3 and

1'. $E(\overline{G'}) = \{ab, cd, ad\}$.

It now follows that

$$Q_1 = R(\mathsf{Ma}(a,b;c,d)) \cup \{G + ad : G \in R(\mathsf{Ma}(a,b;c,d))\},$$

and that

$$M_1 := \{(G + ad, G) : G \in R(\mathsf{Ma}(a,b;c,d))\}$$

is an acyclic perfect matching on $D(Q_1)$.

Now $Q(a,b,c,d) \setminus \bigcup_{i=1}^3 Q_i$ is a union of the pairwise disjoint order convex subposets $R(H)$, where H is one of the three graphs $\mathsf{Cy}(a,b,c,d)$, $\mathsf{Cy}(a,b,d,c)$ or $\mathsf{Cy}(a,c,b,d)$ or one of the nine graphs on vertex set $\{a,b,c,d\}$ which are isomorphic to Pa_3 and not one of $\mathsf{Pa}(c,a,d,b)$, $\mathsf{Pa}(d,a,c,b)$ or $\mathsf{Pa}(b,a,d,c)$. These posets satisfy condition $(\mathcal{P})$ of the Cluster Lemma. So, we will now produce acyclic matchings on the posets $R(H)$, where H is isomorphic to one of the graphs described in Proposition 5.4, other than Ma_2. For any such graph H, define

$$\Delta(H) := \{N_G(1) \cap V(H) : G \in R(H)\}.$$

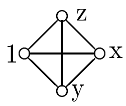
Then $\Delta(H)$ is a simplicial complex and $D(R(H))$ is isomorphic to $D(\Delta(H))$ via the map $G \mapsto N_G(1) \cap V(H)$. Thus a Morse matching on $D(\Delta(H))$ can be lifted to a Morse matching on $D(R(H))$. We will describe Morse matchings for each possible isomorphism type of H. Each of the claims made without proof during these descriptions can be verified by observation, as the complexes $\Delta(H)$ are quite small. Note that if $G \in R(H)$ then any proper $(n-3)$-coloring χ of G satisfies the conditons

- exactly $|V(H)| - 2$ colors are used to color $V(H)$, and $\chi(1)$ is one of these colors, and
- all of the remaining $n - |V(H)| - 1$ colors are used to color the remaining $n - |V(H)| - 1$ vertices of G'.

The promised Morse matchings are described below.

- Say $H = \mathsf{Cy}(x, y, z)$ and $G \in R(H)$. Then every proper $(n-3)$-coloring of G is obtained by assigning one color to $1, x, y$ and z and all of the remaining $n-4$ colors to the remaining vertices of G'. Therefore, $N_G(1) \cap \{x, y, z\} = \emptyset$ and the only face in $\Delta(H)$ is the empty face. The assoicated acyclic matching on $D(R(H))$ is the empty matching, with one critical point, namely the graph G defined by

$$E(\overline{G}) := \{1x, 1y, 1z, xy, xz, yz\}.$$

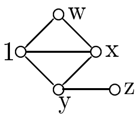


  There are $\binom{n-1}{3}$ such graphs G in $\Gamma_n^t \setminus (\mathcal{K}(n,t) \cup \mathcal{L}(n,t))$.
- Say $H = \mathsf{Pa}(w, x, y, z)$ with $w < z$, and $G \in R(H)$. Then every proper $(n-3)$-coloring of G is obtained by assigning one color γ to w and x, another color γ' to y and z, the remaining $n-5$ colors to the elements of $V(G) \setminus \{1, w, x, y, z\}$ and giving vertex 1 either color γ or γ'. It follows that the maximal elements of $\Delta(H)$ are $\{w, x\}$ and $\{y, z\}$. Now

$$\{(\{w\}, \emptyset), (\{w, x\}, \{x\}), (\{y, z\}, \{y\})\}$$

  is a Morse matching on $\Delta(H)$ whose unique critical point is the 0-simplex $\{z\}$. The unique critical point of the associated acyclic matching on $D(R(H))$ is the graph G defined by

$$E(\overline{G}) := \{1w, 1x, 1y, wx, xy, yz\}.$$



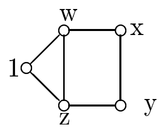
  There are $12\binom{n-1}{4}$ posets $R(\mathsf{Pa}(w, x, y, z))$ to be considered. However, $3\binom{n-1}{4}$ of them are contained in the posets Q_i described above. Therefore, there are $9\binom{n-1}{4}$ such graphs G in $\Gamma_n^t \setminus (\mathcal{K}(n,t) \cup \mathcal{L}(n,t))$ which are critical points of the matchings just described.
- Say $H = \mathsf{Cy}(w, x, y, z)$ with $\min\{w, x, y, z\} = w$, and $G \in R(H)$. Then every proper $(n-3)$-coloring χ of G is obtained by coloring w, x, y, z with two colors γ, γ' so that $\chi(w) \neq \chi(y)$ and $\chi(x) \neq \chi(z)$, coloring the remaining $n-5$ vertices in $V(G')$ with all of the remaining $n-5$ colors and then giving

vertex 1 color γ or γ'. It follows that the maximal faces of $\Delta(H)$ are $\{w,x\}$, $\{w,z\}$, $\{x,y\}$ and $\{y,z\}$ and that

$$\{(\{w\},\emptyset),(\{w,x\},\{x\}),(\{w,z\},\{z\}),(\{y,z\},\{y\})\}$$

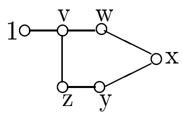
is a Morse matching on $D(\Delta(H))$ whose unique critical point is the 1-simplex $\{x,y\}$. The unique critical cell of the associated acyclic matching on $D(R(H))$ is the graph G defined by

$$E(\overline{G}) := \{1w, 1z, wx, xy, yz, wz\}.$$



There are $3\binom{n-1}{4}$ such graphs G.

- Say $H = \mathsf{Pa}(a,b) + \mathsf{Pa}(x,y,z)$ and $G \in R(H)$. Then every proper $(n-3)$-coloring of G is obtained by coloring a and b with some color γ, coloring x, y and z with two additional colors δ, δ' so that x and z do not get the same color, coloring the remaining $n-6$ vertices in $V(G')$ with the remaining $n-6$ colors and then giving vertex 1 one of the colors γ, δ, δ'. It follows that the maximal faces of $\Delta(H)$ are $\{a,b,x,y\}$, $\{a,b,y,z\}$ and $\{x,y,z\}$. Define
$$\Delta_y(H) := \{\sigma \in \Delta(H) : y \notin \sigma\}.$$
Every maximal face in $\Delta(H)$ contains vertex y, so
$$M(H) := \{(\sigma \cup \{y\}, \sigma) : \sigma \in \Delta_y(H)\}$$
is an acyclic perfect matching on $D(\Delta(H))$. Therefore, $D(R(H))$ admits an acyclic perfect matching.
- Say $H = \mathsf{Pa}(v,w,x,y,z)$ with $v < z$, and $G \in R(H)$. Then any proper $(n-3)$-coloring χ of G is obtained by assigning three distinct colors β, γ, δ to v, x, z respectively, assigning one of β, γ to w and one of γ, δ to y so that $\chi(w) \neq \chi(y)$, assigning all of the remaining $n-6$ colors to the remaining $n-6$ vertices in $V(G')$ and assigning one of β, γ, δ to vertex 1. It follows that the maximal faces of $\Delta(H)$ are $\{v,w,x,y\}$, $\{v,w,y,z\}$ and $\{w,x,y,z\}$. Now every maximal face contains y, and we can argue as we did in the case just above to show that $D(R(H))$ admits an acyclic perfect matching.
- Say $H = \mathsf{Cy}(v,w,x,y,z)$ with $\min\{v,w,x,y,z\} = v$ and $G \in R(H)$. Then every proper $(n-3)$-coloring of G is obtained by assigning three colors β, γ, δ to $\{v,w,x,y,z\}$ so as to obtain a proper 3-coloring of $C(v,x,z,w,y)$, assigning all of the remaining $n-6$ colors to the remaining $n-6$ vertices in $V(G')$, and assigning one of the colors β, γ, δ to vertex 1. It follows that the maximal faces of $\Delta(H)$ are $\{v,w,x,y\}$, $\{v,w,x,z\}$, $\{v,w,y,z\}$, $\{v,x,y,z\}$ and $\{w,x,y,z\}$, so $\Delta(H)$ is the boundary of the simplex $\{v,w,x,y,z\}$. Set
$$\Delta^v(H) := \{\sigma \in \Delta(H) : v \in \sigma\}.$$
Then
$$M(H) := \{(\sigma, \sigma \setminus \{v\}) : \sigma \in \Delta^v(H)\}$$
is a Morse matching on $D(\Delta(H))$ whose unique critical point is the 3-simplex $\{w,x,y,z\}$. The unique critical point of the associated acyclic matching on $D(R(H))$ is the graph G defined by
$$E(\overline{G}) := \{1v, vw, wx, xy, yz, vz\}.$$



There are $12\binom{n-1}{5}$ such graphs G.

- Say $H = \mathsf{Pa}(a,b,c) + \mathsf{Pa}(x,y,z)$ and $G \in R(H)$. Then every proper $(n-3)$-coloring χ of G is obtained by assigning two colors γ, γ' to $\{a,b,c\}$ so that $\chi(a) \neq \chi(c)$, assigning two additional colors δ, δ' to $\{x,y,z\}$ so that $\chi(x) \neq \chi(z)$, assigning all of the remaining $n-7$ colors to the remaining $n-7$ vertices in $V(G')$ and assigning one of the colors $\gamma, \gamma', \delta, \delta'$ to vertex 1. It follows that the maximal faces of $\Delta(H)$ are $\{a,b,c,x,y\}$, $\{a,b,c,y,z\}$, $\{a,b,x,y,z\}$ and $\{b,c,x,y,z\}$. Again, every maixmal face of $\Delta(H)$ contains vertex y, and it follows that $D(R(H))$ admits an acyclic perfect matching.

All possibilities for $R(H)$ have now been examined, and all the critical points G of all the acyclic matchings that have been obtained satisfy

$$|E(\overline{G})| = 6.$$

There are

$$\binom{n-1}{3} + 9\binom{n-1}{4} + 3\binom{n-1}{4} + 12\binom{n-1}{5} = \binom{n-1}{3} + 12\binom{n}{5}$$

critical points, and the next theorem now follows from the Cluster Lemma.

Theorem 5.5. *Let $n \in \mathbb{N}$ with $n \geq 4$. Then there is a Morse matching on Γ_n^{n-3} whose critical points are $\binom{n-1}{3} + 12\binom{n}{5}$ faces of dimension $\binom{n}{2} - 7$. Therefore, Γ_n^{n-3} has the homotopy type of a wedge of $\binom{n-1}{3} + 12\binom{n}{5}$ spheres of dimension $\binom{n}{2} - 7$.*

6. Remarks

As noted in Section 1, we used computer calculations to show that it is not the case that Γ_n^t has the homotopy type of a wedge of spheres of dimension $n(t-1) - \binom{t}{2} - 1$ for all n, t. Our knowledge of the homology of Γ_n^t when $n \leq 9$ is summarized in the following table. The only pairs (n,t) for which the results are not determined by the Theorems in this paper are $(7,3)$, $(8,3)$ and $(8,4)$.

$n \backslash t$	2	3	4	5	6	7	8
3	β_1=1						
4	β_2=3	β_4=1					
5	β_3=16	β_6=6	β_8=1				
6	β_4=105	β_8=82	β_{11}=10	β_{13}=1			
7	β_5=841	$\beta_{10}^{\dagger}$=1535	β_{14}=272	β_{17}=15	β_{19}=1		
8	β_6=7938	?	$\beta_{17}^{\dagger}$=9396 $\beta_{19}^{*\dagger}$=1	β_{21}=707	β_{24}=21	β_{26}=1	
9	β_7=86311	?	?	?	β_{29}=1568	β_{32}=36	β_{34}=1

Table 1. Table of nonzero Betti numbers for the complex Γ_n^t of all t-colorable graphs on n vertices. All known nonzero Betti numbers occur in dimension $n(t-1) - \binom{t}{2} - 1$ except $*$. The values $\dagger$ have been calculated by computer. We have also computed $\tilde{\chi}(\Gamma_8^3) = 31846$.

As noted in the introduction, many questions remain. Is there a nice generating function (or another good description) for the Euler characteristic of Γ_n^t, similar to the result given in Corollary 1.2? Is "most" of the nontrivial homology of Γ_n^t in dimension $n(t-1) - \binom{t}{2} - 1$? In what dimensions does nontrivial homology occur? Is the homology torsion free? Is Γ_n^t homotopy equivalent to a wedge of spheres? The work on complexes of non-Hamiltonian graphs by Jonsson [Jo], along with our computer calculations with Γ_8^4, indicates that reasonable conjectures about graph complexes arising from computations with small vertex sets are often false.

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