ENUMERATING REPRESENTATIONS IN FINITE WREATH PRODUCTS II: EXPLICIT FORMULAE

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ABSTRACT. We present several contributions to the enumerative theory of wreath product representations developed in a previous paper [12]. One of the key ingredients in the description of the generating functions arising in this context (the exterior function Φ_Γ) is calculated. Building on [12, Theorem 1] and this result we derive explicit formulae for the exponential generating function of the series $\{|\mathrm{Hom}(\Gamma, \mathcal{R}_n)|\}$ in the case where Γ is dihedral or a finite abelian group, and the representation sequence $\{\mathcal{R}_n\}$ is any of $\{H \wr S_n\}$ or $\{H \wr A_n\}$ with a finite group H, or the sequence $\{W_n\}$ of Weyl groups of type D_n. Moreover, we verify a conjecture concerning the asymptotic behavior of the sequence $\{|\mathrm{Hom}(G, W_n)|\}$ for finite groups G in the case when G is dihedral or abelian.

INTRODUCTION

For a group Γ denote by $s_\Gamma(n)$ the number of subgroups of index n in Γ. If Γ is finitely generated or of finite (subgroup) rank then $s_\Gamma(n)$ is finite for all n. There is a well-known connection between the problem of counting finite index subgroups in a group Γ and the enumeration of Γ-actions on finite sets, i.e., permutation representations of Γ, manifesting itself in a variety of results scattered throughout the literature. Perhaps the earliest result in this direction is M. Hall's recursion formula [6, Theorem 5.2]

$$s_{F_r}(n) = n(n!)^{r-1} - \sum_{k=1}^{n-1} \big((n-k)!\big)^{r-1} s_{F_r}(k)$$

for the number of index n subgroups in the free group of (finite) rank r. Dey's corresponding formula [4, Theorem 6.10] for (nontrivial) free products is another

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relevant example, as is the identity

$$\sum_{n=0}^{\infty} \frac{|Hom(G,S_n)|}{n!} z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d} z^d\right), \quad m = |G| \tag{1}$$

for a finite group G, which exhibits the exponential generating function of the sequence $\{|\mathrm{Hom}(G,S_n)|\}_0^\infty$ as a particular type of entire function. The most general expression of this correspondence to be found in the literature appears to be [5, Prop. 1], which incorporates the examples mentioned as well as many other results of a similar nature.

More generally, let Γ be a group, H a finite group, and let $\{\Pi_n\}_0^\infty$ denote either the sequence $\{A_n\}_0^\infty$ of alternating groups or the sequence $\{S_n\}_0^\infty$ of symmetric groups. Given $\{\Pi_n\}, H$, and a normal subgroup $N \trianglelefteq H$ with H/N abelian put

$$\big(H \wr \Pi_n\big)_N := \Big\{(f,\pi) \in H \wr \Pi_n :\ \prod_i f(i) \in N\Big\}, \quad n \geq 0.$$

Special cases of this construction include in particular the symmetric wreath products $H \wr S_n$, the alternating wreath products $H \wr A_n$, and the Weyl groups $W_n = (C_2 \wr S_n)_{\mathbf{1}}$ of type D_n. In [12] the exponential generating function of the sequence $\{|\mathrm{Hom}(\Gamma, (H \wr \Pi_n)_N)|\}_{n=0}^\infty$ and of certain refinements of it is computed whenever $(H : N) \leq 2$ and Γ satisfies some rather mild finiteness conditions (always met for instance if Γ is finitely generated). This result [12, Theorem 1], or, rather, the special case (Theorem 1.1) explained in Subsection 1.1 below, will be the starting point for our present investigations.

One of the key ingredients in the description of the generating function

$$\sum_{n\geq 0} |Hom(\Gamma, (H \wr \Pi_n)_N)|\, z^n/n! \tag{2}$$

given in Theorem 1.1 is the *exterior function* Φ_Γ, a certain power series, whose variables $z_{(\Gamma_\Pi,\Gamma_N)}$ are indexed by pairs of subgroups (Γ_Π, Γ_N) of index at most 2 in Γ. With each such subgroup pair (Γ_Π, Γ_N) there is associated a power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$, and the generating function (2) is obtained from the series Φ_Γ by replacing each variable $z_{(\Gamma_\Pi,\Gamma_N)}$ with the corresponding power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$. Our first main result (Theorem 1.3) provides an explicit formula for the exterior function Φ_Γ as an arithmetic mean of certain exponential functions.

In Sections 2 and 3 we apply the theory of Section 1 to derive explicit formulae for the generating functions $\sum_{n=0}^\infty |\mathrm{Hom}(\Gamma, \mathcal{R}_n)|\, z^n/n!$, where Γ is dihedral or finite abelian, and the representation sequence $\{\mathcal{R}_n\}$ is any of $\{H \wr S_n\}$ or $\{H \wr A_n\}$ with a finite group H, or $\{W_n\}$. While the results on dihedral groups are a nice and fairly direct illustration of the general theory, the corresponding analysis for abelian groups is considerably more demanding. Our first task is to determine the orbits of $Aut(G)$ on the subgroups of index 2 in a finite abelian group G. Using duality theory this determination is made in Subsection 3.1 by examining orbits

of $Aut(G)$ on the involutions of G. The next five subsections concentrate on the case when G is a finite abelian 2-group. In Subsection 3.2 we derive identities expressing the generating functions for $|\mathrm{Hom}(G, H \wr A_n)|$ and $|\mathrm{Hom}(G, W_n)|$ in terms of those for $|\mathrm{Hom}(G, H \wr S_n)|$ and $|\mathrm{Hom}(G, C_2 \wr S_n)|$, respectively, and the corresponding inner functions $\Theta_{G_\Pi}^{G_N}(z)$; cf. Proposition 3.17. The inner functions occurring in these identities are calculated explicitly in the next three subsections, and in Subsection 3.7 the results obtained for finite abelian 2-groups are extended to the case of a finite abelian group. In Subsection 3.6 we discuss, by ways of an example, the case when G is a homogeneous abelian 2-group. The formulae associated with the wreath product representations of such a group take a rather pleasant form and serve well to illustrate the results obtained in Section 3 up to that point. Moreover, the special case when G is elementary abelian provides the basis for a fast algorithm to compute the Euler characteristic of the Quillen complex for groups of the form $H \wr S_n$, $H \wr A_n$, and W_n. This aspect will be treated in a separate paper [10].
It has been conjectured in [12] that the asymptotic formula

$$|\mathrm{Hom}(G, W_n)| \sim \big(1 + s_G(2)\big)^{-1} |\mathrm{Hom}(G, C_2 \wr S_n)|$$

holds for an arbitrary finite group G with a relative error of size $\mathcal{O}\big(A^{-n^{1/|G|}}\big)$ and some constant $A = A(G) > 1$. The validity of this conjecture would imply in particular that the asymptotic expansion provided by [12, Theorem 4]) for $|\mathrm{Hom}(G, C_2 \wr S_n)|$ carries over without change to $(1 + s_G(2))\,|\mathrm{Hom}(G, W_n)|$. In Section 4, building on the results of the preceding two sections and an asymptotic estimate for the coefficients of entire functions of finite genus developed in [11, Sect. 2], this conjecture is verified in the case when G is abelian or dihedral.

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1. Enumerating wreath product representations of finitely generated groups

1.1. **The basic counting result.** The starting point for our present investigations will be a special case of [12, Theorem 1] which we now explain. In what follows, Γ and H are groups, $|H| < \infty$, N is a normal subgroup of H with H/N

abelian, and $\{\Pi_n\}_0^\infty$ denotes either the sequence $\{A_n\}_0^\infty$ of alternating groups or the sequence $\{S_n\}_0^\infty$ of symmetric groups.

The series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$

Assume that Γ is finitely generated. Given a subgroup Γ' of (finite) index n in Γ, a homomorphism $\chi : \Gamma' \to H$, and a right transversal $\{\gamma_1, \ldots, \gamma_n\}$ for Γ' in Γ, define

$$\Gamma_\chi := \Big\{\gamma \in \Gamma : \prod_j \chi(\gamma_j\gamma\overline{\gamma_j\gamma}^{-1}) \in N\Big\},$$

where $^- : \Gamma \to \{\gamma_1, \ldots, \gamma_n\}$ associates with each element γ the representative $\overline{\gamma}$ of the coset $\Gamma'\gamma$. Consider the map $\psi : \Gamma \to H/N$ sending γ to $\prod_j \chi(\gamma_j\gamma\overline{\gamma_j\gamma}^{-1})N$. Since H/N is abelian the composition of χ with the canonical projection $H \to H/N$ factors through $\Gamma'/[\Gamma',\Gamma']$ inducing a homomorphism $\chi' : \Gamma'/[\Gamma',\Gamma'] \to H/N$, and we have $\psi = \chi' \circ V_{\Gamma\to\Gamma'}$, where $V_{\Gamma\to\Gamma'} : \Gamma \to \Gamma'/[\Gamma',\Gamma']$ is the transfer of Γ into Γ'. Since the transfer is a homomorphism and independent of the choice of representatives the same is true for ψ; consequently, $\Gamma_\chi = \ker(\psi)$ is a normal subgroup of index at most $(H : N)$ in Γ and is independent of the system of representatives used in its definition. If $\Gamma' = \Gamma$ then $\Gamma_\chi = \chi^{-1}(N)$, while for an abelian group Γ and every homomorphism $\chi : \Gamma' \to H$ with $(\Gamma : \Gamma') < \infty$

$$\Gamma_\chi = \Big\{\gamma \in \Gamma : \ \chi(\gamma^{(\Gamma:\Gamma')}) \in N\Big\}.$$

Given a positive integer n and two normal subgroups $\Gamma_\Pi, \Gamma_N \trianglelefteq \Gamma$ define

$$s(n, \Gamma_\Pi, \Gamma_N) := \sum_{\substack{(\Gamma:\Gamma')=n \\ \tau_{\Gamma'}^{-1}(\Pi(\Gamma'\backslash\Gamma))=\Gamma_\Pi}} \Big|\Big\{\chi \in \mathrm{Hom}(\Gamma', H) : \ \ \Gamma_\chi = \Gamma_N\Big\}\Big|,$$

where $\tau_{\Gamma'} : \Gamma \to S(\Gamma'\backslash\Gamma)$ describes the action of Γ by right multiplication on the set $\Gamma'\backslash\Gamma$ of right cosets of Γ' in Γ, and $\Pi(\Gamma'\backslash\Gamma)$ is the subgroup of $S(\Gamma'\backslash\Gamma)$ isomorphic to Π_n. For each pair (Γ_Π, Γ_N) of normal subgroups in Γ we define a formal power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$ by

$$\Theta_{\Gamma_\Pi}^{\Gamma_N}(z) := \sum_{n\geq 1} \frac{|H|^{n-1} s(n, \Gamma_\Pi, \Gamma_N)}{n} z^n.$$

The function Φ_Γ

Assume from now on that $(H : N) \leq 2$. Consider the system of subgroup pairs

$$\mathcal{U}_\Gamma := \Big\{(\Gamma_\Pi, \Gamma_N) : \ \ \Gamma_\Pi, \Gamma_N \leq \Gamma, \ (\Gamma : \Gamma_\Pi) \leq \alpha_\Pi, \ (\Gamma : \Gamma_N) \leq (H : N)\Big\}$$

where

$$\alpha_\Pi := \begin{cases} 1, & \{\Pi_n\} = \{S_n\} \\ 2, & \{\Pi_n\} = \{A_n\}, \end{cases}$$

and fix a set $E \subseteq \Gamma$ which generates Γ as a normal subgroup. With each pair $(\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma$ we associate a (formal) variable $z_{(\Gamma_\Pi,\Gamma_N)}$ and a discrete variable $\nu(\Gamma_\Pi, \Gamma_N)$ taking nonnegative integral values and form the power series

$$\Phi_\Gamma\big(z_{(\Gamma_\Pi,\Gamma_N)};\, (\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma\big) := \sum \prod_{(\Gamma_\Pi,\Gamma_N)} z_{(\Gamma_\Pi,\Gamma_N)}^{\nu(\Gamma_\Pi,\Gamma_N)} / \nu(\Gamma_\Pi, \Gamma_N)!,$$

where the right-hand sum is extended over those maps $\nu : \mathcal{U}_\Gamma \to \mathbb{N}_0$ satisfying the following $2|E|$ conditions:

$$\left\{ \begin{array}{ll} \sum\limits_{e \notin \Gamma_\Pi} \nu(\Gamma_\Pi, \Gamma_N) \equiv 0 \mod \alpha_\Pi, & e \in E \\ \sum\limits_{e \notin \Gamma_N} \nu(\Gamma_\Pi, \Gamma_N) \equiv 0 \mod (H : N), & e \in E. \end{array} \right. \tag{3}$$

Denote by $\mathcal{J}_\Gamma$ the intersection of all subgroups of index $k_0 := \max(\alpha_\Pi, (H : N))$ in Γ. The quotient $\overline{\Gamma} = \Gamma/\mathcal{J}_\Gamma$ is an elementary abelian 2–group of rank $r = \log_2(\sum_{k \le k_0} s_\Gamma(k))$. One can show that Φ_Γ does in fact not depend on the choice of the normal generating system E and that the functions Φ_Γ and $\Phi_{\overline{\Gamma}}$, formed with respect to the same data $\{\Pi_n\}_0^\infty$, H, and N, coincide up to a canonical identification of variables

$$z_{(\Gamma_\Pi,\Gamma_N)} \mapsto z_{(\Gamma_\Pi/\mathcal{J}_\Gamma, \Gamma_N/\mathcal{J}_\Gamma)}, \quad (\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma;$$

cf. [12, Prop. 2].

The result

Theorem 1.1. *Let Γ be a finitely generated group, H a finite group, $N \leq H$ a subgroup of index $(H:N) \leq 2$, and let $\{\Pi_n\}_0^\infty$ denote either the sequence $\{A_n\}_0^\infty$ of alternating groups or the sequence $\{S_n\}_0^\infty$ of symmetric groups. Then*

$$\sum_{n\geq 0} \frac{|Hom(\Gamma, (H \wr \Pi_n)_N|}{n!} z^n = \Phi_\Gamma\big(\Theta_{\Gamma_\Pi}^{\Gamma_N}(z);\ (\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma\big), \tag{4}$$

i.e., the exponential generating function of the sequence $\{|Hom(\Gamma, (H \wr \Pi_n)_N)|\}_{n=0}^\infty$ is obtained from the series Φ_Γ by replacing each variable $z_{(\Gamma_\Pi, \Gamma_N)}$ with the corresponding power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$.

This is the special case of [12, Theorem 1], where Γ is finitely generated, $\Sigma = \emptyset$, $\Lambda = \mathbb{N}$, and $M = \mathbb{N}_0$. If $\Pi_n = S_n$ $(n \geq 0)$ and $N = H$, the generating function (4) takes a particularly simple form. In this case $(H \wr \Pi_n)_N = H \wr S_n$, $\alpha_\Pi = 1$, and $\mathcal{U}_\Gamma = \{(\Gamma, \Gamma)\}$. Dropping the subscript (Γ, Γ) the exterior function becomes $\Phi_\Gamma(z) = \mathrm{e}^z$, and Theorem 1.1 yields the following.

Corollary 1.2. *Let Γ be a finitely generated group, and H a finite group. Then*

$$\sum_{n\geq 0} \frac{|Hom(\Gamma, H \wr S_n)|}{n!} z^n = \exp\left(\sum_{n\geq 1} \frac{|H|^{n-1} s_\Gamma^H(n)}{n} z^n\right) \tag{5}$$

where $s_\Gamma^H(n) := \sum_{(\Gamma:\Gamma')=n} |Hom(\Gamma', H)|$.

For $\Gamma = G$ a finite group of order m the identity (5) becomes

$$\sum_{n=0}^{\infty} \frac{|Hom(G, H \wr S_n)|}{n!} z^n = \exp\big(P_G^H(z)\big), \tag{6}$$

where

$$P_G^H(z) := \sum_{d|m} \frac{|H|^{d-1} s_G^H(d)}{d} z^d = \frac{1}{|H|} \sum_{U \leq G} |\mathrm{Hom}(U,H)|(|H|z)^{(G:U)}/(G:U),$$

which generalizes (1). The special case of (6) where $G = C_m$ is cyclic of order m yields Chigira's result ([3, Theorem 2])

$$\sum_{n=0}^{\infty} \frac{\iota_m(H \wr S_n)}{n!} z^n = \exp\left(\frac{|H|^{d-1} \iota_{m/d}(H)}{d} z^d\right),$$

where, for a finite group B, $\iota_m(B)$ denotes the number of solutions of the equation $x^m = 1$ in B.

1.2. **Calculation of Φ_Γ.** Here we supplement Theorem 1.1 by establishing an explicit formula for the exterior function Φ_Γ. As remarked above the calculation of Φ_Γ can be reduced to the case where Γ is an elementary abelian 2-group of finite rank. It will be convenient to deal with the latter problem in a linear algebra setting.

1.2.1. *Set-up and result.* For a nonnegative integer r let $V = \mathbf{F}_2^r$ be the vector space of dimension r over the field $\mathbf{F}_2$ with two elements, $E = \{e_1, \ldots, e_r\}$ the standard basis of V, and let $\langle \cdot, \cdot \rangle$ be the usual inner product on V, i.e., $\langle u, v\rangle = \sum u_i v_i$. The assignment $v \mapsto v^\perp$ sending a vector $v \in V$ to its complement defines a bijection from V onto the set

$$\mathcal{H}(V) := \Big\{V' \leq V : \operatorname{codim}_V(V') \leq 1\Big\}.$$

Under this bijection the set $\mathcal{U}_\Gamma$, $\Gamma = (V, +)$, is identified with the subspace

$$\mathcal{U}_V = V_{\alpha_\Pi} \oplus V_{(H:N)}$$

of dimension $R = r(\log_2 \alpha_\Pi + \log_2(H : N))$ in $V \oplus V$, where $V_1 := \{\mathfrak{o}\}$ and $V_2 := V$. The standard basis of $\mathcal{U}_V$ is $\mathcal{E} = \mathcal{U}_V \cap ((E \times V_1) \cup (V_1 \times E))$. The inner product on V induces the inner product

$$\langle (u, v), (x, y)\rangle = \langle u, x\rangle + \langle v, y\rangle$$

on $V \oplus V$ (and, hence, on $\mathcal{U}_V$). Denote by $\Phi_V = \Phi_V(z_{(u,v)};\, (u, v) \in \mathcal{U}_V)$ the power series obtained from Φ_Γ by replacing each variable $z_{(\Gamma_\Pi,\Gamma_N)}$ with the corresponding variable $z_{(u,v)}$, where $u^\perp = \Gamma_\Pi$ and $v^\perp = \Gamma_N$.

Theorem 1.3. *We have*

(7)

$$\Phi_V\big(z_{(u,v)};\, (u, v) \in \mathcal{U}_V\big) = 2^{-R} \sum_{(u,v)\in\mathcal{U}_V} \exp\Bigg(\sum_{\langle (x,y),(u,v)\rangle = 0} z_{(x,y)} - \sum_{\langle (x,y),(u,v)\rangle = 1} z_{(x,y)} \Bigg).$$

The proof of Theorem 1.3 proceeds in three steps.

1.2.2. *A first explicit formula for Φ_V.* By definition,

$$\Phi_V = \sum_{(8)} \prod_{(u,v)\in\mathcal{U}_V} z_{(u,v)}^{\nu(u,v)} / \nu(u, v)!,$$

where the right-hand sum is extended over the maps $\nu : \mathcal{U}_V \to \mathbb{N}_0$ such that

(8)
$$\sum_{\langle (x,y),(u,v)\rangle = 1} \nu(x, y) \equiv 0 \quad (2), \quad (u, v) \in \mathcal{E}.$$

For a map $\nu \in \mathbf{F}_2^{\mathcal{U}_V}$ let

$$S = S(\nu) := \Big\{(u,v) \in \mathcal{U}_V :\ \nu(u,v) = 1\Big\}.$$

Viewing (8) as a system of equations over $\mathbf{F}_2$, a map $\nu : \mathcal{U}_V \to \mathbf{F}_2$ is a solution of (8) if and only if

$$\sum_{(u,v)\in S} (u,v) = (0,0).$$

Hence, introducing the sets

$$\mathcal{S} := \Big\{S \subseteq \mathcal{U}_V :\ \sum_{(u,v)\in S} (u,v) = (0,0)\Big\} \quad \text{and} \quad \mathcal{S}_0 := \Big\{S \in \mathcal{S} :\ (0,0) \notin S\Big\}$$

and blowing up the solutions of (8) over $\mathbf{F}_2$ to maps $\nu : \mathcal{U}_V \to \mathbb{N}_0$ we obtain the following description of the function Φ_V:

$$\begin{aligned}\Phi_V &= \sum_{S\in\mathcal{S}} \prod_{(u,v)\in S} \sinh(z_{(u,v)}) \prod_{(u,v)\in\mathcal{U}_V\setminus S} \cosh(z_{(u,v)}) \\ &= \mathrm{e}^{z_{(0,0)}} \sum_{S\in\mathcal{S}_0} \prod_{(u,v)\in S} \sinh(z_{(u,v)}) \prod_{(u,v)\in\mathcal{U}_V^\sharp\setminus S} \cosh(z_{(u,v)}),\end{aligned} \tag{9}$$

where $\mathcal{U}_V^\sharp := \mathcal{U}_V - \{(0,0)\}$. Expanding the hyperbolic functions in terms of exponential functions, (9) can be rewritten as

$$\Phi_V = \mathrm{e}^{z_{(0,0)}} \sum_{\varepsilon\in\{\pm 1\}^{\mathcal{U}_V^\sharp}} C_\varepsilon \exp\left(\sum_{(u,v)\in\mathcal{U}_V^\sharp} \varepsilon(u,v) z_{(u,v)}\right) \tag{10}$$

with coefficients $C_\varepsilon \in \mathbb{Z}\big[\frac{1}{2^{2^R-1}}\big]$, which will be determined in the next step.

1.2.3. *Calculation of the coefficients C_ε.* For a sign function $\varepsilon : \mathcal{U}_V^\sharp \to \{\pm 1\}$ define

$$E_+(\varepsilon) := \Big\{S \in \mathcal{S}_0 :\ |\{(u,v) \in S : \varepsilon(u,v) = -1\}| \equiv 0 \mod 2\Big\}$$

and

$$E_-(\varepsilon) := \mathcal{S}_0 - E_+(\varepsilon).$$

Then

$$\begin{aligned}C_\varepsilon &= 2^{-|\mathcal{U}_V^\sharp|} \sum_{S\in\mathcal{S}_0} (-1)^{|\{(u,v)\in S:\, \varepsilon(u,v)=-1\}|} \\ &= 2^{-(2^R-1)} \big(|E_+(\varepsilon)| - |E_-(\varepsilon)|\big).\end{aligned} \tag{11}$$

Now we claim the following.

$$\text{If} \quad E_-(\varepsilon) \neq \emptyset \quad \text{then} \quad C_\varepsilon = 0. \tag{12}$$

$$|\mathcal{S}_0| = 2^{2^R - R - 1}. \tag{13}$$

Proof of (12). Consider the additive group Λ of the Boolean ring generated by $\mathcal{U}_V^\sharp$, i.e., the collection of all subsets of $\mathcal{U}_V^\sharp$ with symmetric difference as the group operation. Clearly, $\mathcal{S}_0$ is a subgroup of Λ. Fix $S \in E_-(\varepsilon)$. The translation $T \mapsto S\Delta T$ on $\mathcal{S}_0$ associated with S maps $E_+(\varepsilon)$ to $E_-(\varepsilon)$ and $E_-(\varepsilon)$ to $E_+(\varepsilon)$, thus inducing a bijection between $E_+(\varepsilon)$ and $E_-(\varepsilon)$. Hence, we conclude from (11) that $C_\varepsilon = 0$ in this case.

Proof of (13). Since $\mathcal{U}_V$ is a subspace of $V \oplus V$, $\sum_{(u,v)\in S}(u,v) \in \mathcal{U}_V$ for every $S \subseteq \mathcal{U}_V^\sharp$, and, hence

$$\sum_{(u_0,v_0)\in\mathcal{U}_V} \left|\left\{S \subseteq \mathcal{U}_V^\sharp : \sum_{(u,v)\in S}(u,v) = (u_0,v_0)\right\}\right| = 2^{|\mathcal{U}_V^\sharp|}. \tag{14}$$

Fix $(u_0,v_0) \in \mathcal{U}_V^\sharp$ and define maps

$$\mathcal{S}_0 \underset{\beta_{(u_0,v_0)}}{\overset{\alpha_{(u_0,v_0)}}{\rightleftarrows}} \left\{S \subseteq \mathcal{U}_V^\sharp : \sum_{(u,v)\in S}(u,v) = (u_0,v_0)\right\}$$

as follows:

$$\alpha_{(u_0,v_0)}(S) := \begin{cases} S \cup \{(u_0,v_0)\}, & (u_0,v_0) \notin S \\ S - \{(u_0,v_0)\}, & (u_0,v_0) \in S, \end{cases}$$

$$\beta_{(u_0,v_0)}(S) := \begin{cases} S - \{(u_0,v_0)\}, & (u_0,v_0) \in S \\ S \cup \{(u_0,v_0)\}, & (u_0,v_0) \notin S. \end{cases}$$

Since $\alpha_{(u_0,v_0)}$ and $\beta_{(u_0,v_0)}$ are obviously inverse to each other equation (14) takes the form

$$|\mathcal{U}_V||\mathcal{S}_0| = 2^{|\mathcal{U}_V|-1},$$

whence (13).

Combining (10), (11), (12), and (13) we find that

$$\Phi_V = 2^{-R}\, \mathrm{e}^{z_{(o,o)}} \sum_{\substack{\varepsilon\in\{\pm1\}^{\mathcal{U}_V^\sharp} \\ E_+(\varepsilon)=\mathcal{S}_0}} \exp\Bigg(\sum_{(u,v)\in\mathcal{U}_V^\sharp} \varepsilon(u,v)\, z_{(u,v)}\Bigg). \tag{15}$$

1.2.4. *Characterization of admissible sign functions.* Call a sign function $\varepsilon : \mathcal{U}_V^\sharp \to \{\pm 1\}$ *admissible* if $E_+(\varepsilon) = \mathcal{S}_0$, i.e., if

$$|R(\varepsilon) \cap S| \equiv 0 \quad (2) \quad \text{for all} \quad S \in \mathcal{S}_0, \tag{16}$$

where $R(\varepsilon) := \{(u,v) \in \mathcal{U}_V^\sharp : \varepsilon(u,v) = -1\}$. Clearly, the trivial sign function $\varepsilon = 1$ is admissible. We will show that a sign function $\varepsilon : \mathcal{U}_V^\sharp \to \{\pm 1\}$ is admissible if and only if it is obtained from some $H \in \mathcal{H}(\mathcal{U}_V)$ by setting

$$\varepsilon_H(u,v) = \begin{cases} +1, & (u,v) \in H - \{(\mathsf{o},\mathsf{o})\} \\ -1, & (u,v) \in \mathcal{U}_V - H. \end{cases}$$

This characterization when combined with formula (15) yields (7). Since our claim is obviously true if $\dim(\mathcal{U}_V) \leq 1$, we will assume for the remainder of the proof that $|\mathcal{U}_V| > 2$.

Let ε be an admissible sign function with $R(\varepsilon) \neq \emptyset$. Then

$$|R(\varepsilon)| = |\mathcal{U}_V|/2. \tag{17}$$

To see this, fix $(u_1, v_1) \in R(\varepsilon)$ and let $(u,v) \in \mathcal{U}_V^\sharp - \{(u_1,v_1)\}$. The set

$$S = \Big\{(u_1,v_1),\ (u,v),\ (u_1+u, v_1+v)\Big\}$$

is contained in $\mathcal{U}_V^\sharp$ and sums to (o,o), i.e., $S \in \mathcal{S}_0$; and, since ε is admissible, $R(\varepsilon)$ must contain exactly one of the vectors (u,v), (u_1+u, v_1+v), whence (17).

Since $(\mathsf{o},\mathsf{o}) \notin R(\varepsilon)$, (17) immediately implies that $\langle R(\varepsilon)\rangle = \mathcal{U}_V$, i.e., $R(\varepsilon)$ contains a basis of $\mathcal{U}_V$. Suppose that $\mathcal{E} \subseteq R(\varepsilon)$. If $R(\varepsilon)$ were to contain an element (u,v) with $\langle (u,v),(u,v)\rangle = 0$ then for the set

$$S = \Big\{(u,v)\Big\} \cup \Big\{(u',v') \in \mathcal{E} : \langle (u,v),(u',v')\rangle = 1\Big\} \in \mathcal{S}_0$$

the intersection $R(\varepsilon) \cap S = S$ would have odd cardinality, contradicting the admissibility of ε. Hence, if $\mathcal{E} \subseteq R(\varepsilon)$ then

$$R(\varepsilon) \subseteq \Big\{(u,v) \in \mathcal{U}_V : \langle (u,v),(u,v)\rangle = 1\Big\},$$

and by (17) we must have equality; i.e., $R(\varepsilon)$ is the (nontrivial) coset of the standard hyperplane

$$H_1 = \Big\{(u,v) \in \mathcal{U}_V : \langle (u,v),(u,v)\rangle = 0\Big\}$$

in $\mathcal{U}_V$.

Next, observe that $\mathrm{Aut}(\mathcal{U}_V)$ acts on the set of nontrivial admissible sign functions via $\alpha\cdot\varepsilon := \varepsilon\circ\alpha^{-1}$ and that $R(\alpha\cdot\varepsilon) = \alpha(R(\varepsilon))$. It follows now that the collection of all admissible sign functions is contained in the set $\{\varepsilon_H : H \in \mathcal{H}(\mathcal{U}_V)\}$. Conversely, each sign function ε_H is admissible. Indeed, let H be a hyperplane of $\mathcal{U}_V$ and let S be a subset of $\mathcal{U}_V$. Then we have

$$\sum_{(u,v)\in S\cap(\mathcal{U}_V\backslash H)} (u,v) \in H \quad \text{if and only if} \quad |S\cap(\mathcal{U}_V\backslash H)| \equiv 0 \quad (2)$$

and hence

$$\sum_{(u,v)\in S} (u,v) \in H \quad \text{if and only if} \quad |R(\varepsilon_H)\cap S| \equiv 0 \quad (2);$$

in particular ε_H is admissible. The proof of Theorem 1.3 is thus complete. □

Remark. In order to apply Theorem 1.3 one has to correctly assign the inner functions $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$ to the variables $z_{(u,v)}$ of Φ_V. This is trivial for $\dim(\mathcal{U}_V) \leq 2$, but may be a nontrivial task for $\dim(\mathcal{U}_V) > 2$; compare for instance Subsection 3.2.

2. Dihedral groups

As a first application of the results in Section 1 we provide explicit formulae for the generating functions $\sum_{n=0}^{\infty} |\mathrm{Hom}(D,\mathcal{R}_n)|\, z^n/n!$, where $D = D_l$ is a dihedral group with $l \in \mathbb{N}\cup\{\infty\}$, and the representation sequence $\{\mathcal{R}_n\}$ is any of $\{H \wr S_n\}$ or $\{H \wr A_n\}$ with a finite group H, or $\{W_n\}$.

2.1. **Symmetric representations and a reduction.** For a positive integer l consider the dihedral group

$$D = D_l = \left\langle w, \sigma \mid w^l = \sigma^2 = 1,\ \sigma w \sigma = w^{-1}\right\rangle$$

of order $m = 2l$, and let d be a divisor of m. If $d \mid l$ then D contains d dihedral subgroups of order m/d, namely the groups

$$U_i = \langle w^d,\ w^i\sigma\rangle, \quad 0 \leq i < d.$$

Also, if d is even, D has one cyclic subgroup of index d, namely $U = \langle w^{d/2}\rangle$. Hence,

$$s_D^H(d) = \sum_{(D:U)=d} |\mathrm{Hom}(U,H)| = \begin{cases} d\,|\mathrm{Hom}(D_{l/d},H)|; & d\equiv 1 \quad (2) \\ \iota_{m/d}(H); & d \equiv 0 \quad (2),\ d\nmid l \\ \iota_{m/d}(H) + d\,|\mathrm{Hom}(D_{l/d},H)|; & d\equiv 0 \quad (2),\ d \mid l, \end{cases}$$

and, by (6)

(18)
$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(D_l, H \wr S_n)|}{n!} z^n = \exp\Bigg(\frac{1}{|H|} \sum_{d|l} |\mathrm{Hom}(D_{l/d}, H)| (|H|z)^d + \frac{1}{2|H|} \sum_{d|l} \iota_{l/d}(H) (|H|z)^{2d}/d \Bigg), \quad l \in \mathbb{N}.$$

Next, consider the infinite dihedral group

$$D = D_\infty = \big\langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1 \big\rangle = \big\langle w, \sigma \mid \sigma^2 = 1,\ \sigma w \sigma = w^{-1} \big\rangle,$$

where $w = \sigma\tau$. For each positive integer n this group contains n dihedral subgroups of index n, namely the groups

$$U_i = \langle w^n,\ w^i \sigma \rangle, \quad 0 \leq i < n.$$

Also, if n is even, D contains one cyclic subgroup of index n, namely $U = \langle w^{n/2} \rangle$. Hence,

$$s_D^H(n) = n\,(\iota_2(H))^2 + \begin{cases} |H|, & n \equiv 0 \quad (2) \\ 0, & n \equiv 1 \quad (2) \end{cases}$$

and, by Corollary 1.2

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(D_\infty, H \wr S_n)|}{n!} z^n = \frac{\exp\left(\frac{(\iota_2(H))^2 z}{1 - |H| z} \right)}{\sqrt{1 - (|H|z)^2}}. \tag{19}$$

The function Φ_{D_l} occurring in the enumeration of alternating and Weyl group representations of dihedral groups is given by formula (7) with $(\mathcal{U}_V, \langle \cdot, \cdot \rangle) \cong (V, \langle \cdot, \cdot \rangle)$ and $R = r = 1$ or $= 2$, depending on whether l is odd or $l \in 2\mathbb{N} \cup \{\infty\}$. Note also that in the latter case the two dihedral subgroups of index 2 in D_l are equivalent under a group automorphism. Consequently, by Theorem 1.1

(20)
$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(D_l, H \wr A_n)|}{n!} z^n = \frac{1}{4} \Bigg(\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(D_l, H \wr S_n)|}{n!} z^n \Bigg) \times \Big\{ 1 + 2\mathrm{e}^{-2(\Theta_1(l) + \Theta_2(l))} + \mathrm{e}^{-4\Theta_2(l)} \Big\}, \quad l \in \mathbb{N} \cup \{\infty\},$$

where $\Theta_1(l) := \Theta^{D_l}_{\langle w\rangle}(z)$ and $\Theta_2(l) := \Theta^{D_l}_{\langle w^2,\sigma\rangle}(z)$ or $\Theta_2(l) := 0$, depending on whether $l \in 2\mathbb{N} \cup \{\infty\}$ or l is odd. Similarly, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(D_l, W_n)|}{n!} z^n &= \frac{1}{4}\Bigg(\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(D_l, C_2 \wr S_n)|}{n!} z^n\Bigg) \\ &\quad \times \Big\{1 + 2\mathrm{e}^{-2(\Theta^1(l)+\Theta^2(l))} + \mathrm{e}^{-4\Theta^2(l)}\Big\}, \quad l \in \mathbb{N} \cup \{\infty\},\end{aligned} \tag{21}$$

where $\Theta^1(l) := \Theta^{\langle w\rangle}_{D_l}(z)$ and $\Theta^2(l) := \Theta^{\langle w^2,\sigma\rangle}_{D_l}(z)$ or $\Theta^2(l) := 0$, depending on whether $l \in 2\mathbb{N} \cup \{\infty\}$ or l is odd. Thus, in order to complete the explicit enumeration of alternating and Weyl group representations of dihedral groups it remains to compute the functions $\Theta_1(l)$, $\Theta_2(l)$, $\Theta^1(l)$, and $\Theta^2(l)$. This is done in the next two subsections.

2.2. **The functions $\Theta_i(l)$.** In determining these functions the crucial point is to discuss the invariant $\tau_U^{-1}(A(U\backslash D))$ of a subgroup $U \leq D$:
(i) Let $l < \infty$, $k \mid l$, and let $U = \langle w^k\rangle$ be the cyclic subgroup of index $d = 2k$ in $D = D_l$. A system of representatives for $U\backslash D$ is given by the elements

$$1,\, w,\, w^2, \ldots, w^{k-1},\, \sigma,\, \sigma w,\, \sigma w^2, \ldots, \sigma w^{k-1}.$$

The generator w acts on $U\backslash D$ as the product $(1, w, \ldots, w^{k-1})(\sigma, \sigma w, \ldots, \sigma w^{k-1})$ of two k-cycles, while σ acts as the product $(w, \sigma w^{k-1})(w^2, \sigma w^{k-2}) \ldots (w^{k-1}, \sigma w)(1, \sigma)$ of k transpositions. Consequently, $\langle w\rangle \leq \tau_U^{-1}(A(U\backslash D))$, and $\sigma \in \tau_U^{-1}(A(U\backslash D))$ if and only if k is even.
(ii) Let $l < \infty$, $d \mid l$, $0 \leq i < d$, and let $U = \langle w^d, w^i\sigma\rangle$ be the i^{th} dihedral subgroup of index d in $D = D_l$. Now $U\backslash D$ can be represented by the elements $1, w, w^2, \ldots, w^{d-1}$, and w acts on $U\backslash D$ as the d-cycle $(1, w, \ldots, w^{d-1})$; i.e., $w \in \tau_U^{-1}(A(U\backslash D))$ if and only if d is odd. If d is odd then σ has precisely one fixed point on $U\backslash D$, if d is even then σ has 0 or 2 fixed points, depending on whether i is odd or even. Hence, $\sigma \in \tau_U^{-1}(A(U\backslash D))$ if and only if one of the following holds:

$$d \equiv 1 \quad (4), \quad d \equiv 0 \quad (4) \;\text{ and }\; i \equiv 1 \quad (2), \quad d \equiv 2 \quad (4) \;\text{ and }\; i \equiv 0 \quad (2).$$

(iii) If $l = \infty$ and $U = \langle w^k\rangle$ is the cyclic subgroup of index $n = 2k$ in $D = D_\infty$ or if $U = \langle w^n, w^i\sigma\rangle$ is the i^{th} dihedral subgroup of index n in D then essentially the same discussion applies as in (i) respectively (ii).

From these observations we find that

$$\begin{aligned}
\Theta_1(l) &= \frac{1}{|H|}\Big\{\sum_{\substack{d\mid l\\ d\equiv 3(4)}} |\mathrm{Hom}(D_{l/d},H)|(|H|z)^d + \frac{1}{2}\sum_{\substack{d\mid l\\ d\equiv 1(2)}} \iota_{l/d}(H)(|H|z)^{2d}/d\Big\}, \quad l\in\mathbb{N},\\
\Theta_1(\infty) &= \frac{(|H|\iota_2(H))^2z^3}{1-(|H|z)^4} + \log\Big(\frac{1}{\sqrt{1-(|H|z)^2}}\Big) + \log\Big(\sqrt[4]{1-(|H|z)^4}\Big),\\
\Theta_2(2k) &= \frac{1}{2|H|}\sum_{d\mid k} |\mathrm{Hom}(D_{k/d},H)|(|H|z)^{2d}, \quad k\in\mathbb{N},\\
\Theta_2(\infty) &= \frac{(\iota_2(H))^2|H|z^2}{2(1-(|H|z)^2)}.
\end{aligned}$$

2.3. **The functions $\Theta^i(l)$.** Here the crucial point is to identify the invariant D_χ attached to a homomorphism $\chi: U\to C_2$, where U is a subgroup of finite index in $D=D_l$.

(i) If $l<\infty$, $k\mid l$, $U=\langle w^k\rangle$ is the cyclic subgroup of index $d=2k$ in $D=D_l$, and $\chi: U\to C_2$ is a homomorphism, then the equations

$$\begin{aligned}
&\prod_{j=0}^{k-1}\chi(w^{j+1}\overline{w^{j+1}}^{-1})\prod_{j=0}^{k-1}\chi(\sigma w^{j+1}\overline{\sigma w^{j+1}}^{-1}) = \chi(w^k)\chi(w^{-k}) = 1\\
&\quad\prod_{j=0}^{k-1}\chi(w^j\sigma\overline{w^j\sigma}^{-1})\prod_{j=0}^{k-1}\chi(\sigma w^j\sigma\overline{\sigma w^j\sigma}^{-1}) = \prod_{j=1}^{k-1}\chi(w^j\sigma w^{j-k}\sigma)\prod_{j=1}^{k-1}\chi(\sigma w^j\sigma w^{j-k})\\
&\qquad\qquad = (\chi(w^k))^{k-1}(\chi(w^{-k}))^{k-1} = 1
\end{aligned}$$

show that $D_\chi = D$. Of course, $\mathrm{Hom}(U,C_2)$ has one or two elements depending on whether l/k is odd or even.

(ii) Let $l<\infty$, $d\mid l$, $0\le i<d$, $U=\langle w^d, w^i\sigma\rangle$ the i^{th} dihedral subgroup of index d in $D=D_l$, and let $\chi: U\to C_2$ be a homomorphism. We have

$$\prod_{j=0}^{d-1}\chi(w^{j+1}\overline{w^{j+1}}^{-1}) = \chi(w^d)$$

and

$$\begin{aligned}
\prod_{j=0}^{d-1}\chi(w^j\sigma\overline{w^j\sigma}^{-1}) &= \prod_{j=0}^{i}\chi(w^j\sigma w^{j-i})\prod_{j=i+1}^{d-1}\chi(w^j\sigma w^{j-i-d})\\
&= (\chi(w^i\sigma))^d(\chi(w^d))^{d-i-1}.
\end{aligned}$$

Now we distinguish two cases:
(a) If $l/d \equiv 1\ (2)$ then we must have $\chi(w^d) = 1$, i.e., $D_\chi \geq \langle w \rangle$. Apart from the trivial homomorphism $\chi_0 \equiv 1$ (with $D_{\chi_0} = D$) there is one other homomorphism $\chi_1 : U \to C_2$, and we have

$$D_{\chi_1} = \begin{cases} \langle w \rangle, & d \equiv 1 \quad (2) \\ D, & d \equiv 0 \quad (2). \end{cases}$$

(b) If $l/d \equiv 0\ (2)$ then there are four homomorphisms $\chi : U \to C_2$:

$$\begin{aligned}
&\chi_0 \equiv 1 : \quad D_{\chi_0} = D, \\
&\chi_w \quad \text{with} \quad \chi_w(w^d) \neq 1,\ \chi_w(w^i\sigma) = 1 : \quad D_{\chi_w} = \begin{cases} \langle w^2, \sigma \rangle, & d \equiv i+1 \quad (2) \\ \langle w^2, w\sigma \rangle, & \text{otherwise}, \end{cases} \\
&\chi_\sigma \quad \text{with} \quad \chi_\sigma(w^d) = 1,\ \chi_\sigma(w^i\sigma) \neq 1 : \quad D_{\chi_\sigma} = \begin{cases} \langle w \rangle, & d \equiv 1 \quad (2) \\ D, & d \equiv 0 \quad (2), \end{cases} \\
&\chi_{w,\sigma} \quad \text{with} \quad \chi_{w,\sigma}(w^d) \neq 1 \neq \chi_{w,\sigma}(w^i\sigma) : \quad D_{\chi_{w,\sigma}} = \begin{cases} \langle w^2, \sigma \rangle, & i \equiv 1 \quad (2) \\ \langle w^2, w\sigma \rangle, & i \equiv 0 \quad (2). \end{cases}
\end{aligned}$$

(iii) If $l = \infty$ and $U = \langle w^k \rangle$ is the cyclic subgroup of index $n = 2k$ in $D = D_\infty$ or if $U = \langle w^n, w^i\sigma \rangle$ is the i^{th} dihedral subgroup of index n in D then there are two respectively four homomorphisms $\chi : U \to C_2$, and the discussion of (i) or (ii)(b) applies, respectively.
These observations now yield the following explicit expressions for the functions $\Theta^i(l)$:

$$\begin{aligned}
\Theta^1(l) &= \frac{1}{2} \sum_{\substack{d \mid l \\ d \equiv 1(2)}} (2z)^d, \quad l \in \mathbb{N}, \\
\Theta^1(\infty) &= \frac{z}{1-(2z)^2}, \\
\Theta^2(2k) &= \frac{1}{2} \Bigg\{ \sum_{\substack{d \mid k \\ d \equiv 1(2)}} (2z)^d + \sum_{\substack{d \mid k \\ k/d \equiv 0(2)}} (2z)^{2d} \Bigg\}, \quad k \in \mathbb{N}, \\
\Theta^2(\infty) &= \frac{z}{1-2z}.
\end{aligned}$$

3. Abelian Groups

In this section, we provide explicit formulae for the generating functions $\sum_{n=0}^{\infty} |\mathrm{Hom}(G, \mathcal{R}_n)|\, z^n/n!$, where G is a finite abelian group and the series $\{\mathcal{R}_n\}$ is any of $\{H \wr S_n\}$ or $\{H \wr A_n\}$ for a fixed finite group H, or $\{W_n\}$. We will use basic notation from the theory of partitions, as found in [7], which we recall below.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_j, \dots)$ be a partition, that is, a sequence of nonnegative integers λ_j such that $\lambda_j \geq \lambda_{j+1}$ for each $j \in \mathbb{N}$ and only finitely many λ_j are nonzero. If $\lambda_r > 0$ and $\lambda_{r+1} = 0$, then we also write $\lambda = (\lambda_1, \dots, \lambda_r)$. For each $i \in \mathbb{N}$, the multiplicity $m_i(\lambda)$ of i in λ is the number of parts of λ having size i, that is,

$$m_i(\lambda) := |\,\{j \in \mathbb{N} : \lambda_j = i\}\,|.$$

We sometimes write m_i for $m_i(\lambda)$ when it is clear which partition λ is being considered. The weight $|\lambda|$ of a partition λ is the sum of its parts, that is,

$$|\lambda| := \sum_{i \geq 1} \lambda_i.$$

The conjugate λ' of λ is the partition whose parts are defined by

$$\lambda'_i := \sum_{j \geq i} m_j(\lambda), \quad i \geq 1.$$

In particular, $\lambda'_1 = \sum_{j \geq 1} m_j(\lambda)$ is sometimes called the length or norm of λ. A partial ordering $\subseteq$ is defined on the set of all partitions by

$$\mu \subseteq \lambda :\Leftrightarrow \mu_i \leq \lambda_i \text{ for all } i.$$

Definition 3.1. *Let p be a prime and let λ be a partition. Then $G(p, \lambda)$ is the finite abelian p-group defined by*

$$G(p, \lambda) := \bigoplus_{j \geq 1} C_{p^{\lambda_j}} = \bigoplus_{i \geq 1} m_i(\lambda) C_{p^i}.$$

If $G = G(p, \lambda)$ with $\lambda = (\lambda_1, \dots, \lambda_r)$ and $E = \{e_1, \dots, e_r\}$ is a basis for G, we will always assume that $|e_j| = p^{\lambda_j}$ for all $j \in [r]$. Furthermore, if $g \in G$ is written as $g = \sum_{j=1}^{r} \alpha_j e_j$, we will always assume that $0 \leq \alpha_j < |e_j|$ for all j.

P. Hall determined, for any three partitions λ, μ, ν, a polynomial $g^{\lambda}_{\mu,\nu}(t)$ such that for every prime p, the number of subgroups $H \leq G \cong G(p, \lambda)$ with $H \cong G(p, \mu)$ and $G/H \cong G(p, \nu)$ is $g^{\lambda}_{\mu,\nu}(p)$. The reader may refer to [7, Chapter II], for the main results on these Hall polynomials, which will be important in what follows.

For a group G, we denote by $\mathcal{L}(G)$ the lattice of all subgroups of G, ordered by inclusion.

3.1. **Orbits of Aut(G) on minimal and maximal subgroups.** An essential ingredient in constructing the explicit formulae promised above is the determination of the orbits of Aut(G) on subgroups of index two in G. We will use the duality results given immediately below in order to make this determination by examining orbits of Aut(G) on subgroups of G of order two.

Definition 3.2. *Let G be a finite abelian group with basis $\{e_1, \dots, e_m\}$. Let $g = \sum_{j=1}^{m} \alpha_j e_j \in G$, where $\alpha_j \in \mathbb{N}$. The character $\chi_g : G \to \mathbb{C}^*$ is the homomorphism defined by*

$$e_j \mapsto \exp(\frac{2\pi i \alpha_j}{|e_j|}), \quad 1 \le j \le m.$$

The following result is well known (see for example [13], Lemma 8.1.3).

Lemma 3.3. *Let G be a finite abelian group. The map $\phi : G \to Hom(G, \mathbb{C}^*)$ defined by $g \mapsto \chi_g$ is a group isomorphism, where multiplication in $Hom(G, \mathbb{C}^*)$ is pointwise.*

Definition 3.4. *Let G be a finite abelian group. For $H \le G$, define*

$$H^* := \{g \in G : \chi_g(H) = 1\}.$$

Lemma 3.5 (Baer, [1])**.** *Let G be a finite abelian group.*

1. *The map on $\mathcal{L}(G)$ determined by $H \mapsto H^*$ is an antiautomorphism of $\mathcal{L}(G)$.*
2. *For all $H \le G$, we have $(H^*)^* = H$.*

Proof. The first claim is essentially Theorem 8.1.4 in [13]. We sketch the proof here, including some facts which will be used later. For $H \le G$, define

$$\hat{H} := \{\chi \in \mathrm{Hom}(G, \mathbb{C}^*) : H \le \ker(\chi)\}$$

and for $X \le \mathrm{Hom}(G, \mathbb{C}^*)$ set $\ker(X) := \bigcap_{\chi \in X} \ker(\chi)$. By Lemma 3.3, it is sufficient to establish the following three facts

- The map from $\mathcal{L}(G)$ to $\mathcal{L}(\mathrm{Hom}(G, \mathbb{C}^*))$ determined by $H \mapsto \hat{H}$ is order reversing.
- The map from $\mathcal{L}(\mathrm{Hom}(G, \mathbb{C}^*))$ to $\mathcal{L}(G)$ defined by $X \mapsto \ker(X)$ is order reversing.
- For all $H \le G$ we have $\ker(\hat{H}) = H$.

The first two facts follow immediately from the definitions. To prove the third, first note that $H \le \ker(\hat{H})$. Let $g \in G - H$. Since $[G/H, G/H] = 1$, there exists some $\overline{\chi} \in \mathrm{Hom}(G/H, \mathbb{C}^*)$ such that $\overline{\chi}(g + H) \ne 1$. Now $\overline{\chi}$ determines a map $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$ such that $H \le \ker(\chi)$ and $\chi(g) = \overline{\chi}(g + H) \ne 1$. Thus

$H = \ker(\hat{H})$ as claimed.
To prove the second claim, we first note that by the definition of χ_g, we have $\chi_g(h) = \chi_h(g)$ for all $g, h \in G$. This gives $H \leq (H^*)^*$. We now proceed by induction on $\mathcal{L}(G)^{\mathrm{op}}$. If $H = G$ then by the first claim of the lemma we have $(H^*)^* = 1^* = H$. Now assume for contradiction that $H < K := (H^*)^* \leq G$. Using the first claim of the lemma twice, we must have $K^* < H^*$ and $K = (H^*)^* < (K^*)^*$, contradicting our inductive hypothesis. □

Corollary 3.6. *Let G be a finite abelian group and let $U \leq G$. Then $U^* \cong G/U$.*

Proof. We first prove the claim of the corollary when G is a p-group for some prime p. By Lemma 3.5, the interval $[U, G]$ is isomorphic to the lattice $\mathcal{L}(U^*)^{\mathrm{op}}$, where $X \leq Y$ in $\mathcal{L}(U^*)^{\mathrm{op}}$ if and only if $Y \leq X$ in $\mathcal{L}(U^*)$. Applying Lemma 3.5(1) to $\mathcal{L}(U^*)$, we get $\mathcal{L}(U^*)^{\mathrm{op}} \cong \mathcal{L}(U^*)$. Thus G/U and U^* are finite abelian p-groups with isomorphic subgroup lattices. The claim in this case now follows from a result of Baer (see [2]) which also appears as Theorem 2.6.8 in [13].
Now we turn to the general case. For an arbitrary finite abelian group H and a prime p, we denote by $P_p(H)$ the Sylow p-subgroup of H. Let π be the set of all primes, so $G = \oplus_{p\in\pi} P_p(G)$. If $H \leq G$ is a p-group for some $p \in \pi$, we denote by H^{*_p} the dual of H (that is, H^*) in $P_p(G)$.
Let $\{e_1, \ldots, e_r\}$ be a basis for G such that each e_j has prime power order. Let $g = \sum_{j=1}^{r} \gamma_j e_j \in G$. We may (and do) assume that $0 \leq \gamma_j < |e_j|$ for each j. Thus if $|g|$ is a power of the prime p and p does not divide $|e_j|$ then $\gamma_j = 0$, so $\chi_g(e_j) = \chi_{e_j}(g) = 1$. It now follows from the fact that $H^* = \cap_{h\in H} \ker(\chi_h)$ that if $H \leq G$ is a p-group for some $p \in \pi$ then

$$H^* = H^{*_p} \oplus \bigoplus_{q\in\pi\setminus\{p\}} P_q(G).$$

Thus (again using the description of H^* as an intersection of kernels) we see that if $U = \oplus_{p\in\pi} P_p(U) \leq G$ then

$$U^* = \bigoplus_{p\in\pi} (P_p(U))^{*_p}.$$

By the result of the corollary in the case when $|G|$ is a prime power, we have

$$U^* \cong \bigoplus_{p\in\pi} P_p(G)/P_p(U).$$

This last direct sum is isomorphic to G/U. □

Our next goal is to define, for an endomorphism α of a finite abelian group G, an endomorphism which is an adjoint to α in the appropriate sense.

Definition 3.7. *Let G be a finite abelian group and let $\{e_1, \dots, e_m\}$ be a basis for G. Let $\alpha \in End(G)$.*

- *A matrix of α with respect to the given basis is any matrix $(\alpha_{ij}) \in M_m(\mathbb{Z})$ satisfying $\alpha(e_j) = \sum_{k=1}^{m} \alpha_{jk} e_k$ for $1 \leq j \leq m$.*
- *If (α_{jk}) is a matrix for α, then α^* is the endomorphism of G such that the matrix (α^*_{kj}) defined by*
$$\alpha^*_{kj} := \alpha_{jk} \frac{|e_j|}{|e_k|}$$
is a matrix for α^* with respect to the given basis.

Since $|\alpha(e_j)|$ divides $|e_j|$ for all j, α^*_{kj} is an integer for all pairs j, k. Also, if (α_{jk}) is a fixed matrix for α with respect to a given basis, then the set of all matrices for α with respect to that basis consists of all $(\beta_{jk}) \in M_m(\mathbb{Z})$ such that $|e_k|$ divides $\alpha_{jk} - \beta_{jk}$ for all pairs j, k. Therefore α^* is well defined. A straightforward matrix calculation shows that for $\alpha, \beta \in \mathrm{End}(G)$ we have $(\alpha\beta)^* = \beta^* \alpha^*$. In particular, for $\alpha \in \mathrm{Aut}(G)$, we have $(\alpha^{-1})^* = (\alpha^*)^{-1}$.

Lemma 3.8. *Let G be a finite abelian group and let $\alpha \in Aut(G)$. Then*

1. *$\alpha^* \in Aut(G)$.*
2. *For all $g, h \in G$, $\chi_{g^\alpha}(h) = \chi_g(h^{\alpha^*})$.*

Proof. Let $\{e_1, \dots, e_m\}$ be a basis for G. Since α normalizes each Sylow subgroup of G and G is the direct sum of its Sylow subgroups, we may assume in the proof of the first claim that G is a p-group for some prime p. It is known that for any $\beta \in End(G)$, we have $\beta \in \mathrm{Aut}(G)$ if and only if the determinant of (β_{jk}) is not divisible by p (see [14], Satz 2). Let $A \in M_m(\mathbb{Q})$ be the matrix obtained from $(\alpha_{jk})^T$ by multiplying row k by $\frac{1}{|e_k|}$ for each k. Then (α^*_{jk}) is obtained from A by multiplying column j by $|e_j|$ for each j. Thus (α_{jk}) and (α^*_{jk}) have the same determinant, and the first claim follows. To prove the second claim, it suffices to show that the claimed equality holds when $g = e_j$ and $h = e_k$ for arbitrary $j, k \in [m]$, and the equality in these cases follows immediately from the definitions. ☐

Corollary 3.9. *Let G be a finite abelian group and let $H, K \leq G$. Then H and K are conjugate under the action of $Aut(G)$ if and only if H^* and K^* are conjugate under that action. In particular, if $(G : H) = (G : K) = 2$, then H and K are conjugate if and only if the involutions generating H^* and K^* are conjugate.*

Proof. It follows from Lemma 3.8 that for any $g \in G$ and $\alpha \in \mathrm{Aut}(G)$ we have $\ker(\chi_{g^\alpha}) = \ker(\chi_g)^{(\alpha^*)^{-1}}$. It was established in the proof of Lemma 3.5 that $H =$

$\bigcap_{g \in H^*} \ker(\chi_g)$. Given these two facts, and the fact that $(\alpha^{-1})^* = (\alpha^*)^{-1}$, it is straightforward to show that if $H^\alpha = K$ then $(H^*)^{\alpha^*} = K^*$. The particular statement follows from Corollary 3.6. □

With the desired duality results in hand, we now investigate the action of $\mathrm{Aut}(G)$ on involutions in G when G is a finite abelian 2-group. We work in a slightly more general setting, investigating the action of $\mathrm{Aut}(G)$ on subgroups of G of order p when G is a finite abelian p-group. We will make use of the following well known result (see, for example, [8, Chap. XI, Prop. 3].

Lemma 3.10. *Let G be a finite abelian group and let X be a cyclic subgroup of G such that the order of X is the exponent of G. Then there is a complement to X in G. In particular, there exists a basis for G which includes a generator of X.*

For the remainder of this subsection, p is a prime, $G = G(p, \lambda)$ is a finite abelian p-group of type λ, and $\{e_1, \ldots, e_r\}$ is a basis for G.

Definition 3.11. *Let $X \leq G$ with $|X| = p$. Define*

$$m(X) := \max\{|U| : X \leq U \leq G, \, U \, cyclic\}.$$

Lemma 3.12. *Let $X \leq G$ with $|X| = p$ and let $x = \sum_{j=1}^r \alpha_j e_j$ generate X. Then*

$$m(X) = \min\left\{p^{\lambda_i} : \alpha_i \neq 0\right\}.$$

Proof. Let $U \leq G$ be cyclic such that $X \leq U$ and $|U| = m(X)$. There exists some generator $u = \sum_{j=1}^r \beta_j e_j$ of U such that $\frac{m(X)}{p} u = x$. If $p^{\lambda_j} < m(X)$ then $\frac{m(X)}{p} \beta_j \equiv 0 \bmod p^{\lambda_j}$. On the other hand, if $\alpha_j = 0$ whenever $\lambda_j < \lambda_i$ then $p^{\lambda_i - 1}$ divides α_j for all j. Let $\gamma_j = p^{1-\lambda_i}\alpha_j$, and set $u = \sum_{j=1}^r \gamma_j e_j$. Then $|u| = p^{\lambda_i}$ and $p^{\lambda_i - 1} u = x$. □

Lemma 3.13. *Let $X, Y \leq G$ with $|X| = |Y| = p$. Then X and Y are conjugate under the action of $Aut(G)$ if and only if $m(X) = m(Y)$.*

Proof. First assume that there is some $\alpha \in \mathrm{Aut}(G)$ such that $X^\alpha = Y$. Then α maps each cyclic subgroup of G containing X to a cyclic subgroup of G containing Y, and it follows immediately that $m(X) = m(Y)$.
Now assume that there exist cyclic subgroups $T, U \leq G$ such that $X \leq T$, $Y \leq U$ and $|T| = m(X) = m(Y) = |U|$. Set $K = \langle e_j : |e_j| > m(X) \rangle$. Note that every maximal cyclic subgroup of K has order greater than $m(X)$. Thus $T \cap K = 0$, since otherwise we would have $X \leq K$ and $m(X) > |T|$. Let $\overline{G} = G/K$, and let

$\overline{T}$ be the image of T under the natural projection from G to $\overline{G}$. Then $\overline{T} \cong T$, and $|\overline{T}|$ is the exponent of $\overline{G}$. By Lemma 3.10, there is a complement $\overline{S}$ to $\overline{T}$ in $\overline{G}$. Let S be the full preimage of $\overline{S}$ in G. It is straightforward to show that $G = S \oplus T$. By the same argument, there is a complement to U in G. It follows that there is an automorphism of G which maps T to U. □

Corollary 3.14. *For $i \in \mathbb{N}$, let $\mathcal{M}_i = \{X \leq G : |X| = p$ and $m(X) = p^i\}$. Then*

$$|\mathcal{M}_i| = \frac{p^{\lambda'_i} - p^{\lambda'_{i+1}}}{p-1}.$$

Proof. Since a group of prime order p has $p-1$ generators, $|\mathcal{M}_i|$ can be obtained by dividing by $p-1$ the number of elements of order p in G which generate a subgroup X satisfying $m(X) = p^i$. Let $x = \sum_{j=1}^{r} \alpha_j e_j \in G$ with $|x| = p$. Since $|x| = p$, $p^{\lambda_j - 1}$ divides α_j for all j. Let $X = \langle x \rangle$. By Lemma 3.12, $m(X) = p^i$ if and only if $\alpha_j = 0$ whenever $\lambda_j < i$ and there exists some j such that $\lambda_j = i$ and $\alpha_j \neq 0$. There are $(p^{m_i(\lambda)} - 1)\prod_{k>i} p^{m_k(\lambda)} = p^{\lambda'_i} - p^{\lambda'_{i+1}}$ ways to choose such an x. □

3.2. **Correspondence of index two subgroups and hyperplanes.** Here we will describe an explicit bijection between the maximal subgroups of a finite abelian 2-group G and the nonzero elements of $G/\Phi(G)$. This will allow us to correctly assign the functions $\Theta^G_{G_\Pi}$ or $\Theta^{G_N}_G$ to the variables of the formula given in Theorem 1.3 when $\mathcal{R}_n$ is $H \wr A_n$ or W_n. As a result, we obtain a fairly simple formula expressing the generating function for $|\mathrm{Hom}(G, H \wr A_n)|$ in terms of that for $|\mathrm{Hom}(G, H \wr S_n)|$ and the inner functions $\Theta^G_{G_\Pi}$, along with an analogous formula expressing the generating function for $|\mathrm{Hom}(G, W_n)|$ in terms of that for $|\mathrm{Hom}(G, C_2 \wr S_n)|$ and functions $\Theta^{G_N}_G$. **For the remainder of this subsection,** $G = G(2, \lambda)$ is a finite abelian 2-group with basis $\{e_1, \dots, e_r\}$.

We begin by stating some elementary facts. Note that $\{e_1 + \Phi(G), \dots, e_r + \Phi(G)\}$ is a basis for $G/\Phi(G)$. From now on, we will write $\overline{e_i}$ for $e_i + \Phi(G)$. Let $t = \sum_{j=1}^{r} \tau_j e_j \in G$ with $0 \leq \tau_j < |e_j|$ for each j. Then t is an involution if and only if $\tau_j \in \left\{0, \frac{|e_j|}{2}\right\}$ for each j, and some τ_j is nonzero.
In order to describe the desired bijection between the maximal subgroups of G and the nonzero elements of $G/\Phi(G)$, we will define a correspondence between the minimal subgroups of these two groups and examine the relation between duality in G and orthogonality in $G/\Phi(G)$.

Definition 3.15. *Let $t = \sum_{j=1}^{r} \tau_j e_j \in G$ be an involution.*

- $S(t) := \{ j \in [r] : \tau_j = 2^{\lambda_j - 1} \}$.
- $v(t) := \sum_{j \in S(t)} \overline{e_j} \in G/\Phi(G)$.

If we think of t and $v(t)$ as vectors with respect to the bases $\{e_j\}$ and $\{\overline{e_j}\}$, then $v(t)$ is the vector over $\mathbf{F}_2$ which has zero entries exactly where t has zero entries.

Lemma 3.16. *Let $\langle .,. \rangle$ be the usual inner product on $G/\Phi(G)$ with respect to the basis $\overline{e_1}, \ldots, \overline{e_r}$. Let M be a maximal subgroup of G and let $t \in G$ be the involution generating M^*. Then*

$$M/\Phi(G) = \Big\{ x \in G/\Phi(G) : \langle v(t), x \rangle = 0 \Big\}.$$

Proof. Write $t = \sum_{j=1}^{r} \tau_j e_j$. By definition, for $g = \sum_{j=1}^{r} \alpha_j e_j$ we have

$$\chi_g(t) = \chi_t(g) = (-1)^{\sum_{j \in S(t)} \alpha_j} = (-1)^{|\{j \in S(t) : \alpha_j \equiv 1(2)\}|}.$$

Thus $g \in M$ if and only if $|\{j \in S(t) : \alpha_j \equiv 1(2)\}|$ is even. Now note that $\Phi(G) = \{2g : g \in G\}$. Therefore

$$g + \Phi(G) = \sum_{\alpha_j \equiv 1(2)} \overline{e_j},$$

and

$$\langle g + \Phi(G), v(t) \rangle \equiv |\{j \in S(t) : \alpha_j \equiv 1(2)\}| \bmod 2.$$

The lemma follows immediately. □

Proposition 3.17. *For each i such that $m_i(\lambda) \neq 0$, let $T_i \leq G$ be a subgroup of order two with $m(T_i) = 2^i$. Let $M_i = T_i^*$ and set $\Theta_i(z) = \Theta_{M_i}^{G}(z)$ and $\Theta^i(z) = \Theta_G^{M_i}(z)$. For all $i \in \mathbb{N}$ such that $m_i(\lambda) = 0$, set $\Theta_i(z) = \Theta^i(z) = 0$. For a given finite group H, write $\mathcal{S}_{G,H}(z)$ for $\sum_{n \geq 0} \frac{|Hom(G, H \wr S_n)|}{n!} z^n$, and $\mathcal{A}_{G,H}(z)$ for $\sum_{n \geq 0} \frac{|Hom(G, H \wr A_n)|}{n!} z^n$. Write $\mathcal{W}_G(z)$ for $\sum_{n \geq 0} \frac{|Hom(G, W_n)|}{n!} z^n$. For $a \in \mathbb{N}$ define*

$$C_a(z) := \sum_{i<a} (2^{\lambda'_i} - 2^{\lambda'_{i+1}}) \Theta_i(z) + 2^{\lambda'_a} \Theta_a(z),$$

and

$$C^a(z) := \sum_{i<a} (2^{\lambda'_i} - 2^{\lambda'_{i+1}}) \Theta^i(z) + 2^{\lambda'_a} \Theta^a(z).$$

Then for any finite group H we have

$$\mathcal{A}_{G,H}(z) = \frac{\mathcal{S}_{G,H}(z)}{2^r} \left[1 + \sum_{a \in \mathbb{N}} (2^{\lambda'_1 - \lambda'_{a+1}} - 2^{\lambda'_1 - \lambda'_a}) e^{-C_a(z)} \right],$$

where r is the rank of G. Also,

$$\mathcal{W}_G(z) = \frac{\mathcal{S}_{G,C_2}(z)}{2^r}\left[1 + \sum_{a\in\mathbb{N}}(2^{\lambda'_1-\lambda'_{a+1}} - 2^{\lambda'_1-\lambda'_a})e^{-C^a(z)}\right].$$

Proof. We will prove the formula for $\mathcal{A}_{G,H}(z)$. The proof of the formula for $\mathcal{W}_G(z)$ is essentially the same. Let V be a vector space of dimension r over $\mathbf{F}_2$ with basis $\{\overline{e_1},\dots,\overline{e_r}\}$, and let $\langle .,.\rangle$ be the usual inner product on V with respect to this basis. For $i\in\mathbb{N}$ such that $m_i(\lambda)>0$ define

$$V_i := \big\langle \overline{e_j} : \lambda_j \geq i \big\rangle,$$

and

$$i^+ := \begin{cases} \min\big\{k > i : m_k(\lambda) > 0\big\}, & i \neq \lambda_1 \\ \infty, & i = \lambda_1. \end{cases}$$

Note that $|V_i| = 2^{\lambda'_i}$. By Theorems 1.1 and 1.3 and Lemmas 3.12 and 3.16, we have

$$\mathcal{A}_{G,H}(z) = 2^{-r}\sum_{v\in V}\exp\Big(\sum_{x\in v^\perp}\Theta_x - \sum_{x\in V-v^\perp}\Theta_x\Big),$$

where $\Theta_{\mathfrak{o}} = \Theta^G_G$, and, for $x \in V - \{\mathfrak{o}\}$, $\Theta_x = \Theta_i$ if and only if $x \in V_i - V_{i^+}$. Let $V^\sharp = V - \{\mathfrak{o}\}$. It follows from Corollary 3.6 and Lemmas 3.12, 3.13, and 3.16 that as x runs through V, Θ_x runs through all Θ^G_M such that $(G:M)\leq 2$. Thus it follows from Corollary 1.2 and the definition of the Θ^G_M that

$$\mathcal{S}_{G,H}(z) = \exp\Big(\sum_{x\in V}\Theta_x\Big).$$

Therefore,

$$\mathcal{A}_{G,H}(z) = \frac{\mathcal{S}_{G,H}(z)}{2^r}\left[1 + \sum_{v\in V^\sharp}\exp\Big(-2\sum_{x\in V-v^\perp}\Theta_x\Big)\right].$$

For fixed $v \in V^\sharp$, we get

$$2\sum_{x\in V-v^\perp}\Theta_x = \sum_{m_i(\lambda)>0} a_i(v)\Theta_i,$$

where

$$\begin{aligned} a_i(v) &= 2(|V_i - v^\perp| - |V_{i^+} - v^\perp|) \\ &= \begin{cases} 2^{\lambda'_i} - 2^{\lambda'_{i+1}}; & V_{i^+} \not\leq v^\perp \\ 2^{\lambda'_i}; & V_{i^+} \leq v^\perp, V_i \not\leq v^\perp \\ 0; & V_i \leq v^\perp. \end{cases} \end{aligned}$$

Note that $V_i \leq v^\perp$ if and only if $v \in V_i^\perp = \langle \overline{e_j} : \lambda_j < i \rangle$. Write $v = \sum_{j=1}^r \alpha_j \overline{e_j}$ and define

$$a(v) := \max \Big\{ \lambda_j : \alpha_j \neq 0 \Big\}.$$

Then $v \notin V_i^\perp$ if and only if $i \leq a(v)$, and we get

$$2 \sum_{x \in V - v^\perp} \Theta_x = \sum_{i < a(v)} (2^{\lambda'_i} - 2^{\lambda'_{i+1}}) \Theta_i(z) \, + \, 2^{\lambda'_{a(v)}} \Theta_{a(v)}.$$

For $a \in \mathbb{N}$, the number of $v \in V^\sharp$ such that $a(v) = a$ is $2^{\lambda'_1 - \lambda'_{a+1}} - 2^{\lambda'_1 - \lambda'_a}$, and we are done. $\square$

3.3. **The generating function for Hom**$(G(p,\lambda), H \wr S_n)$**.** By Corollary 1.2 we have for any two finite groups G, H that

$$\sum_{n \geq 0} \frac{|Hom(G, H \wr S_n)|}{n!} z^n = \exp \Big(\sum_{d \in \mathbb{N}} \frac{|H|^{d-1} s_G^H(d)}{d} z^d \Big),$$

where

$$s_G^H(d) = \sum_{(G:U)=d} |\mathrm{Hom}(U, H)|.$$

We will make this generating function more explicit when $G = G(p,\lambda)$ by giving a formula for $s_G^H(d)$. Note that for two groups G, H with G finite and any $d \in \mathbb{N}$ we have $s_G^H(d) = 0$ unless d divides $|G|$. Thus if $G = G(p,\lambda)$ we are interested in the case when d is a power of p. In this case, a subgroup of G having index $d = p^\rho$ is isomorphic to $G(p,\mu)$ for some μ such that $|\lambda| - |\mu| = \rho$. We make use of a known formula which enumerates subgroups in $G(p,\lambda)$ of a given isomorphism type (see [9]). Note that this formula also gives the number of quotients of G of the same isomorphism type (this is well known and follows from Corollary 3.6). Recall that for a prime power q, the q-binomial coefficient $\binom{n}{k}_q$ is defined to be the number of k-dimensional subspaces in an n-dimensional vector space over the field with q elements. It is well known that

$$\binom{n}{k}_q = \prod_{j=0}^{k-1} \frac{q^{n-j} - 1}{q^{k-j} - 1}.$$

Proposition 3.18 (Miller, [9])**.** *Let λ, μ be partitions and let p be a prime. Then $G(p,\lambda)$ contains a subgroup isomorphic to $G(p,\mu)$ if and only if $\mu \subseteq \lambda$. If $\mu \subseteq \lambda$ then the number of subgroups of $G(p,\lambda)$ isomorphic to $G(p,\mu)$ is*

$$\prod_{i \geq 1} p^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \binom{\lambda'_i - \mu'_{i+1}}{\mu'_i - \mu'_{i+1}}_p.$$

Corollary 3.19. *Let $G = G(p, \lambda)$ be a finite abelian p-group and let H be any finite group. Let $d \in \mathbb{N}$. Then*

1. *If d is not a power of p then $s_G^H(d) = 0$.*
2. *If $d = p^\rho$ for some integer ρ then*

$$s_G^H(d) = \sum_{\substack{\mu \subseteq \lambda \\ |\lambda| - |\mu| = \rho}} \prod_{i \geq 1} p^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \binom{\lambda'_i - \mu'_{i+1}}{\mu'_i - \mu'_{i+1}}_p |Hom(G(p, \mu), H)|.$$

3.4. **The generating function for Hom**$(G(2, \lambda), H \wr A_n)$**.** Let $G = G(2, \lambda)$ be a finite abelian 2-group and fix any finite group H. Given the main results of the previous two sections, we can obtain an explicit formula for the exponential generating function of the sequence $|\mathrm{Hom}(G, H \wr A_n)|$ by finding explicit formulae for $\Theta_M^G(z)$ for all $M < G$ such that $(G : M) = 2$. By definition, we have

$$\Theta_M^G(z) = \sum_{d \in \mathbb{N}} \frac{|H|^{d-1} s(d, M, G)}{d} z^n,$$

where

$$s(d, M, G) = \sum_{\substack{(G:U) = d \\ \tau_U^{-1}(A(U \backslash G)) = M}} |\mathrm{Hom}(U, H)|.$$

Recall that $\tau_U^{-1}(A(U\backslash G))$ is simply the set of elements of G which act as alternating permutations on the coset space $U\backslash G$. Of course, if d is not a power of two, we have $s(d, M, G) = 0$. Our first step is to fix M and determine which $U \leq G$ satisfy $\tau_U^{-1}(A(U\backslash G)) = M$. This step is straightforward for an arbitrary finite abelian group.

Lemma 3.20. *Let G be a finite abelian group and fix $M < G$ with $[G : M] = 2$. Let $U \leq G$. Then $\tau_U^{-1}(A(U\backslash G)) = M$ if and only if $U \leq M$ and the Sylow 2-subgroup of G/U is cyclic.*

Proof. Since G is abelian, we have $U \trianglelefteq G$. It follows easily that $\tau_U(G) \leq S(U\backslash G)$ is the image of G/U under its regular permutation representation $\rho_U : G/U \to S(U\backslash G)$. Let $n = (G : U)$. If n is odd then the claim of the lemma holds trivially, so we can assume that n is even. If $\overline{x} \in G/U$ has order m, then $\rho_U(\overline{x})$ is the product of $\frac{n}{m}$ m-cycles. Thus $\rho_U(\overline{x}) \notin A(U\backslash G)$ if and only if m is even and n/m is odd. This last condition holds if and only if $\overline{x}$ generates the Sylow 2-subgroup of G/U. It follows that $\tau_U^{-1}(A(U\backslash G)) \neq G$ if and only if the Sylow 2-subgroup of G/U is cyclic. If this last condition holds, G/U has a unique subgroup of index two, so there is a unique subgroup of index two in G which contains U. ☐

The corollary below now follows easily from Corollary 3.6.

Corollary 3.21. *Let $G = G(2,\lambda)$ be a finite abelian 2-group and let M be a maximal subgroup of G. Let $T = M^*$. Let $\mathcal{C}(T)$ be the set of all cyclic subgroups of G containing T. Then for any $d \in \mathbb{N}$ we have*

$$s(d,M,G) = \sum_{\substack{C \in \mathcal{C}(T)\\ |C|=d}} |Hom(G/C,H)|.$$

The remainder of this section will be devoted to finding a more explicit formula for the sum given in Corollary 3.21 for arbitrary d and T. Of course if S,T are minimal subgroups of $G = G(2,\lambda)$ which are conjugate under the action of Aut(G), then $s(d,S^*,G) = s(d,T^*,G)$ for all d. Thus we can examine one representative for each $\mathrm{Aut}(G)$ conjugacy class of minimal subgroups. We saw in Section 3.2 that the class of a minimal subgroup T is determined by $m(T)$, and the size of each class was determined therein. The main point below is that if $U \leq G$ is a nontrivial cyclic subgroup and T is the minimal subgroup of G contained in U then $m(T)$ is explicitly determined by the isomorphism type of G/U. This reduces the problem of giving the desired explicit formula to that of determining, for a fixed partition $\mu \subseteq \lambda$, the number of cyclic subgroups $C \leq G$ of order $d = 2^\rho$ such that $G/C \cong G(2,\mu)$. In other words, we need to know $g^\lambda_{\mu,(\rho)}(2)$ for $\mu \subseteq \lambda$ and $1 \leq \rho \leq \lambda_1$. Although the known explicit formulae for the Hall polynomials $g^\lambda_{\mu,\nu}(t)$ are quite complicated, a relatively simple formula exists when one of the partitions μ,ν has only one part. We begin with the main point.

Definition 3.22. *Let λ,μ be partitions with $\mu \subset \lambda$. Define*

$$i(\mu,\lambda) := \max\Big\{j : m_j(\mu) < m_j(\lambda)\Big\}.$$

Lemma 3.23. *Let C be a nontrivial cyclic subgroup of $G = G(p,\lambda)$ with $G/C \cong G(p,\mu)$. Let T be the minimal subgroup of G contained in C. Then*

$$m(T) = p^{i(\mu,\lambda)}.$$

Proof. Let $\{e_1,\dots,e_r\}$ be a basis for G with $|e_i| = p^{\lambda_i}$. For $j \in \mathbb{N}$, define

$$G_{\geq j} := \big\langle e_k : \lambda_k \geq j\big\rangle \quad \text{and} \quad G_{<j} := \big\langle e_k : \lambda_k < j\big\rangle.$$

First note that for any j we have $T \leq G_{\geq j}$ if and only if $C \cap G_{\geq j} \neq 0$. By Lemma 3.12, $m(T) \geq p^j$ if and only if $T \leq G_{\geq j}$. Thus $m(T) < p^j$ if and only if $C \cap G_{\geq j} = 0$. We can now finish the proof by showing that $C \cap G_{\geq j} = 0$ if and only if $i(\mu,\lambda) < j$. Let $\overline{G} = G/C$, and for $K \leq G$ let $\overline{K}$ be the image of K under the natural projection from G to $\overline{G}$. If $C \cap G_{\geq j} = 0$, then $\overline{G_{\geq j}} \cong G_{\geq j}$. This

implies that $(\lambda_1, \dots, \lambda_{\lambda'_j}) \subseteq \mu$ and, since $\mu \subseteq \lambda$, we must have $m_k(\mu) = m_k(\lambda)$ for all $k \geq j$. Thus $i(\mu, \lambda) < j$.
On the other hand, if $C \cap G_{\geq j} \neq 0$, then there is some $\nu \subset (\lambda_1, \dots, \lambda_{\lambda'_j})$ (note the proper containment) such that $\overline{G_{\geq j}} \cong G(p, \nu)$. Since G is abelian and $\overline{G} = \overline{G_{\geq j}} + \overline{G_{<j}}$, $\overline{G}$ is a homomorphic image of $H = \overline{G_{\geq j}} \oplus \overline{G_{<j}}$. Now $H \cong G(p, \rho)$ for some partition ρ, and for $k \geq j$ we have $m_k(\mu) \leq m_k(\rho) \leq m_k(\nu)$. There is some $k \geq j$ such that $m_k(\nu) < m_k(\lambda)$, so $i(\mu, \lambda) \geq j$. □

Now we state the result for the Hall polynomials $g^{\lambda}_{\mu,(\rho)}(t)$ which is given in [7]. We require some additional definitions from the theory of partitions, all of which appear in the first chapter of [7]. If λ and μ are partitions with $\mu \subseteq \lambda$ then $\lambda - \mu$ is a sequence of nonnegative integers, as is $\lambda' - \mu'$. The sequence $\lambda - \mu$ is called a horizontal ρ-strip if $\lambda'_i - \mu'_i \leq 1$ for all i and $|\lambda| - |\mu| = \rho$. If $\lambda - \mu$ is a horizontal ρ-strip, define

$$I(\lambda, \mu) := \Big\{ i \in \mathbb{N} : \lambda'_i - \mu'_i = 1,\, \lambda'_{i+1} - \mu'_{i+1} = 0 \Big\}.$$

For any partition λ, define

$$n(\lambda) := \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2}.$$

Proposition 3.24. [7, Chap. II, 4.13] *Let λ, μ be partitions with $\mu \subseteq \lambda$, and let $\rho \in \mathbb{N}$. If $\lambda - \mu$ is not a horizontal ρ-strip, then $g^{\lambda}_{\mu,(\rho)}(t) = 0$. If $\lambda - \mu$ is a horizontal ρ-strip, then*

$$g^{\lambda}_{\mu,(\rho)}(t) = \frac{t^{n(\lambda)-n(\mu)}}{1 - t^{-1}} \prod_{i \in I(\lambda,\mu)} \big(1 - t^{-m_i(\lambda)}\big).$$

We can now combine the results we have obtained so far in order to give the desired explicit formula. We will use the notation

$$f_p(\mu, H) := |\mathrm{Hom}(G(p, \mu), H)|,$$

where p is a prime, μ is a partition and H is a finite group.

Definition 3.25. *Let $G = G(2, \lambda)$ be a finite abelian 2-group and let $\rho, i \in \mathbb{N}$. Define*

$$\mathcal{P}(\lambda, i, \rho) := \Big\{ \mu \subseteq \lambda : \lambda - \mu \textit{ is a horizontal } \rho\textit{-strip, and } i(\mu, \lambda) = i \Big\}.$$

Lemma 3.26. *Let M be a maximal subgroup of $G = G(2,\lambda)$ and let ρ be a positive integer. If $m(M^*) = 2^i$ then*

$$s(2^\rho, M, G) = \frac{2^{1+n(\lambda)}}{2^{\lambda'_i} - 2^{\lambda'_{i+1}}} \sum_{\mu \in \mathcal{P}(\lambda,i,\rho)} \frac{1}{2^{n(\mu)}} \prod_{j \in I(\lambda,\mu)} (1 - 2^{-m_j(\lambda)}) f_2(\mu, H).$$

Proof. Let $C(M^*)$ be the number of conjugates of M^* under the action of $\mathrm{Aut}(G)$. By Corollary 3.21 and Lemma 3.23, we have

$$s(2^\rho, M, G) = \frac{1}{C(M^*)} \sum_{\mu \in \mathcal{P}(\lambda,i,\rho)} g^\lambda_{\mu,(\rho)}(2) f_2(\mu, H).$$

The lemma now follows from Corollary 3.14 and Proposition 3.24. □

Corollary 3.27. *Let M be a maximal subgroup of $G = G(2,\lambda)$. If $m(M^*) = 2^i$ then*

$$\Theta^G_M(z) = \frac{2^{1+n(\lambda)}}{|H|(2^{\lambda'_i} - 2^{\lambda'_{i+1}})} \sum_{0<\rho\le\lambda_1} \frac{1}{2^\rho} \sum_{\mu \in \mathcal{P}(\lambda,i,\rho)} \frac{1}{2^{n(\mu)}} \prod_{j \in I(\lambda,\mu)} \frac{2^{m_j(\lambda)} - 1}{2^{m_j(\lambda)}} f_2(\mu, H)(|H|z)^{2^\rho}.$$

Proof. This follows immediately from Lemma 3.26. □

An explicit formula for the generating function of $\{|\mathrm{Hom}(G(2,\lambda), H \wr A_n)|\}_{n=0}^\infty$ can now be obtained by combining the results of Proposition 3.17 and Corollaries 3.19 and 3.27, observing that the function Θ^G_M of Corollary 3.27 is equal to the function Θ_i of Proposition 3.17.

3.5. **The generating function for $|\mathbf{Hom}(G(2,\lambda), W_n)|$.** Proposition 3.17 can be used in the same way it was used in the alternating wreath product case to determine an explicit exponential generating function for $|\mathrm{Hom}(G(2,\lambda), W_n)|$. To do this, we must calculate $\Theta^M_G(z)$ for all maximal $M < G$. By definition,

$$\Theta^M_G(z) = \sum_{d \in \mathbb{N}} \frac{2^{d-1} s(d, G, M)}{d} z^d,$$

where

$$s(d, G, M) = \sum_{(G:U)=d} \left| \left\{ \chi \in \mathrm{Hom}(U, C_2) : \ G_\chi = M \right\} \right|.$$

Here G_χ is as defined in Section 1. When G is abelian, it is easy to determine G_χ.

Lemma 3.28. *Let G be a finite abelian group and let $U \le G$. Let $\chi : U \to C_2$ be a homomorphism with kernel K. Then*

1. *If $K = U$ or if the Sylow 2-subgroup of G/K is not cyclic, then $G_\chi = G$.*
2. *If $(U : K) = 2$ and the Sylow 2-subgroup of G/K is cyclic, then G_χ is the unique subgroup of index two in G which contains K.*

Proof. We will use multiplicative notation. If $K = U$ it follows immediately that $G_\chi = G$, so we can assume that $(U : K) = 2$. Let $T = \{t_1, \dots, t_d\}$ be a right transversal to U in G. For any $g \in G$, the map $t_i \mapsto \overline{t_i g}$ is by definition a permutation of T. Since G is abelian and χ is a homomorphism, this gives

$$\prod_{i=1}^{d} \chi(t_i g \overline{t_i g}^{-1}) = \chi(g^d).$$

Therefore, $G_\chi = \{g \in G : g^d \in K\}$. Let P be the Sylow 2-subgroup of G/K and let Q be the complement to P in G/K. Then $|Q|$ divides d, so for any $\overline{g} \in Q$, we have $\overline{g}^d = 1$. Thus $G_\chi < G$ if and only if there is some $\overline{h} \in P$ such that $\overline{h}^d \neq 1$. Write $d = 2^a b$ with b odd, so $|P| = 2^{a+1}$. Then for $\overline{h} \in P$, we have $\overline{h}^d = 1$ if and only if $\overline{h}^{2^a} = 1$. Therefore there is some $\overline{h} \in P$ with $\overline{h}^d \neq 1$ if and only if P is cyclic. If P is cyclic then G/K contains a unique subgroup of index two, whose full preimage in G is the unique subgroup of index two in G which contains K, and therefore must be G_χ. ☐

Before using this result, we introduce some notation.

Definition 3.29. *Let λ be a partition and let ρ be a nonnegative integer. Then $\lambda - \rho$ is the partition defined by*

$$(\lambda - \rho)_i = \max\{\lambda_i - \rho, 0\}.$$

Corollary 3.30. *Let M be a maximal subgroup of $G = G(2, \lambda)$ and let ρ be a nonnegative integer. Then*

$$s(2^\rho, G, M) = \begin{cases} \frac{|G|}{2^{\rho+|\lambda-\rho|}}, & 2^\rho < m(M^*) \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Note that if $K \leq U \leq G$ with $(U : K) = 2$ then there is a unique homomorphism from U to C_2 with kernel K. By Corollary 3.6 and Lemma 3.28, we have

$$s(2^\rho, G, M) = \left|\{K \leq M : G/K \cong C_{2^{\rho+1}}\}\right| = \left|\{C \leq G : M^* \leq C, C \cong C_{2^{\rho+1}}\}\right|.$$

Let $\{e_1, \dots, e_r\}$ be a basis for G and let t generate M^*. By Lemma 3.12, we may assume that $t = 2^{\lambda_j - 1} e_j$, where $m(M^*) = 2^{\lambda_j}$. If $\rho \geq \lambda_j$ then $s(2^\rho, G, M) = 0$ by the definition of $m(M^*)$. Now assume $\rho < \lambda_j$, and let $M^* \leq C \leq G$ with $C \cong C_{2^{\rho+1}}$. Then there is a unique generator $c = \sum_{i=1}^{r} \gamma_i e_i$ of C such that

$\gamma_j = 2^{\lambda_j-\rho-1}$, and for $i \neq j$ we must have $2^{\lambda_i}|2^{\rho}\gamma_i$. Moreover, any c satisfying these conditions generates a group of order $2^{\rho+1}$ containing t. We now choose such a c. Fix $i \neq j$. If $\lambda_i \geq \rho$ then there are 2^{ρ} ways to choose γ_i, while if $\lambda_i < \rho$ then there are 2^{λ_i} ways to choose γ_i. These choices can be made independently, so there are 2^k ways to choose c, where

$$k = \rho\big(-1 + \sum_{i\geq\rho} m_i(\lambda)\big) + \sum_{i<\rho} i m_i(\lambda) = |\lambda| - (\rho + |\lambda-\rho|).$$

The corollary follows, since $|G| = 2^{|\lambda|}$. □

The next corollary now follows immediately from the definition of $\Theta_G^M(z)$ and easy calculations, and, when combined with Proposition 3.17 and Corollary 3.19, gives an explicit formula for the desired generating function.

Corollary 3.31. *Let M be a maximal subgroup of $G = G(2,\lambda)$ with $m(M^*) = 2^i$. Then*

$$\Theta_G^M(z) = |G| \sum_{\rho=0}^{i-1} \frac{(2z)^{2^\rho}}{2^{1+2\rho+|\lambda-\rho|}}.$$

3.6. **An Example.** Here, we briefly interrupt our train of thought to discuss the case when G is a homogeneous abelian p-group. The formulae associated with the wreath product representations of such a group take a rather pleasant form and serve well to illustrate the results obtained so far in this section. Moreover, the special case when G is elementary abelian will play an important role in [10], which deals with the Quillen complex of groups of the form $H \wr S_n$, $H \wr A_n$, and W_n. The general discussion of finite abelian groups will be resumed and brought to an end in the next subsection.

Let $G = rC_{p^\alpha}$ be the homogeneous abelian p-group of rank r and exponent p^α, and let H be a finite group. The type λ of G is of the form $\lambda = (\alpha^r)$, $\lambda' = (r^\alpha)$, and, by Corollaries 1.2 and 3.19

(22)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, H \wr S_n)|}{n!} z^n =$$

$$\exp\left(\frac{1}{|H|}\sum_{\rho=0}^{\alpha r} p^{-\rho}(|H|z)^{p^\rho} \sum_{\substack{\mu\subseteq(\alpha^r)\\ |\mu|=\alpha r-\rho}} |\mathrm{Hom}(G(p,\mu), H)| \prod_{i=1}^{\alpha} p^{\mu'_{i+1}(r-\mu'_i)} \binom{r-\mu'_{i+1}}{\mu'_i - \mu'_{i+1}}_p\right).$$

For the remainder of this subsection we take $p = 2$, i.e., $G = G(2,\lambda)$ with λ as above. If $M < G$ is a maximal subgroup then $m(M^*) = 2^\alpha$, that is, all maximal subgroups of G are conjugate under the action of $\mathrm{Aut}(G)$. In the context of

Corollary 3.27 we have $\mathcal{P}(\lambda, \alpha, \rho) = \{\mu\}$, where μ is the partition consisting of $r-1$ parts α, and a last part $\alpha - \rho$. Also, for this μ, $I(\lambda, \mu) = \{\alpha\}$. Hence,

$$\Theta_\alpha(z) = \frac{2^{1-r}}{|H|} \sum_{\rho=1}^{\alpha} 2^{\rho(r-2)} |\mathrm{Hom}((r-1)C_{2^\alpha} \oplus C_{2^{\alpha-\rho}}, H)| (|H|z)^{2^\rho},$$

and, by Proposition 3.17

(23)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, H \wr A_n)|}{n!} z^n = 2^{-r} \left(\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, H \wr S_n)|}{n!} z^n \right)$$
$$\times \left\{ 1 + (2^r - 1) \exp\left(- \frac{2}{|H|} \sum_{\rho=1}^{\alpha} 2^{\rho(r-2)} |\mathrm{Hom}((r-1)C_{2^\alpha} \oplus C_{2^{\alpha-\rho}}, H)| (|H|z)^{2^\rho} \right) \right\}.$$

Also, by Corollary 3.31

$$\Theta^\alpha(z) = \frac{1}{2} \sum_{0 \le \rho < \alpha} 2^{\rho(r-2)} (2z)^{2^\rho},$$

and Proposition 3.17 yields

$$\text{(24)} \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, W_n)|}{n!} z^n =& 2^{-r} \left(\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, C_2 \wr S_n)|}{n!} z^n \right) \\ & \times \left\{ 1 + (2^r - 1) \exp\left(- 2^{r-1} \sum_{0 \le \rho < \alpha} 2^{\rho(r-2)} (2z)^{2^\rho} \right) \right\}. \end{aligned}$$

For $\alpha = 1$, that is, G elementary abelian, formulae (22) – (24) simplify further. Reverting to multiplicative notation we find that for $r \ge 0$ and every finite group H

(25)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(C_p^r, H \wr S_n)|}{n!} z^n = \exp\left(\frac{1}{|H|} \sum_{\rho=0}^{r} p^{-\rho} \binom{r}{\rho}_p |\mathrm{Hom}(C_p^{r-\rho}, H)| (|H|z)^{p^\rho} \right),$$

(26)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(C_2^r, H \wr A_n)|}{n!} z^n = 2^{-r} \left(\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(C_2^r, H \wr S_n)|}{n!} z^n \right)$$
$$\times \left\{ 1 + (2^r - 1) \exp\left(- \frac{2^{r-1}}{|H|} |\mathrm{Hom}(C_2^{r-1}, H)| (|H|z)^2 \right) \right\},$$

and
(27)
$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(C_2^r, W_n)|}{n!} z^n = 2^{-r} \left(\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(C_2^r, C_2 \wr S_n)|}{n!} z^n \right) \left\{ 1 + (2^r - 1)\mathrm{e}^{-2^r z} \right\}.$$

3.7. **Extending the results to finite abelian groups.** We now extend the results obtained above for finite abelian 2-groups to finite abelian groups. If G is a finite abelian group and P is the Sylow 2-subgroup of G then $G = P \oplus Q$, where Q is the direct sum of the Sylow subgroups of odd order in G. Every subgroup of index two in G is of the form $M \oplus Q$, where M is a maximal subgroup of P. This immediately gives a correspondence of Φ_G with Φ_P, identifying $z_{(M \oplus Q, M' \oplus Q)}$ with $z_{(M,M')}$ (see [12, Proposition 2]). With this fact in hand, most of the work needed to extend our earlier results to arbitrary finite abelian groups is routine.

Definition 3.32. *Let π be the set of all primes. A **π-partition** $\lambda^{(\pi)}$ is a set of partitions $\left\{\lambda^{(p)} : p \in \pi\right\}$ indexed by the elements of π such that $|\lambda^{(p)}| = 0$ for all but finitely many primes p. We define a partial order $\subseteq_{(\pi)}$ on the set of all π-partitions by $\mu^{(\pi)} \subseteq_{(\pi)} \lambda^{(\pi)}$ if and only if $\mu^{(p)} \subseteq \lambda^{(p)}$ for all $p \in \pi$. For any π-partition $\lambda^{(\pi)}$, the finite abelian group $G(\lambda^{(\pi)})$ is defined as*
$$G(\lambda^{(\pi)}) := \bigoplus_{p \in \pi} G(p, \lambda^{(p)}).$$

We can use Proposition 3.18 in order to generalize Corollary 3.19. We first introduce some notation.

Definition 3.33. *Let p be a prime and let λ, μ be partitions with $\mu \subseteq \lambda$. Define*
$$h_p(\lambda, \mu) := \prod_{i \geq 1} p^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \binom{\lambda'_i - \mu'_{i+1}}{\mu'_i - \mu'_{i+1}}_p.$$
Let $\lambda^{(\pi)}, \mu^{(\pi)}$ be π-partitions with $\mu^{(\pi)} \subseteq_{(\pi)} \lambda^{(\pi)}$. Define
$$j(\lambda^{(\pi)}, \mu^{(\pi)}) := \prod_{p \in \pi} p^{|\lambda^{(p)}| - |\mu^{(p)}|},$$
and for any π-partition $\mu^{(\pi)}$ and any finite group H define
$$f(\mu^{(\pi)}, H) := |Hom(G(\mu^{(\pi)}), H)|.$$

Proposition 3.34. *Let $G = G(\lambda^{(\pi)})$ be a finite abelian group and let H be any finite group. Let $d \in \mathbb{N}$. Then*

1. *If d does not divide $|G|$ then $s_G^H(d) = 0$.*

2. *If d divides $|G|$ then*

$$s_G^H(d) = \sum_{\substack{\mu^{(\pi)} \subseteq_{(\pi)} \lambda^{(\pi)} \\ j(\lambda^{(\pi)}, \mu^{(\pi)}) = d}} f(\mu^{(\pi)}, H) \prod_{p \in \pi} h_p(\lambda^{(p)}, \mu^{(p)}).$$

Proof. This follows from Proposition 3.18. □

It is worth remarking at this point that if H is abelian, then

$$f(\mu^{(\pi)}, H) = \prod_{p \in \pi} f_p(\mu^{(p)}, H),$$

where f_p is as defined before Definition 3.25.

Proposition 3.35. *Let $G = G(\lambda^{(\pi)})$ be a finite abelian group of order m and let H be any finite group. Then*

$$\sum_{n=0}^{\infty} \frac{|Hom(G, H \wr S_n)|}{n!} z^n = \exp\big(P_G^H(z)\big),$$

where

$$P_G^H(z) = \frac{1}{|H|} \sum_{d|m} \frac{1}{d} \sum_{\substack{\mu^{(\pi)} \subseteq_{(\pi)} \lambda^{(\pi)} \\ j(\lambda^{(\pi)}, \mu^{(\pi)}) = d}} f(\mu^{(\pi)}, H) \prod_{p \in \pi} h_p(\lambda^{(p)}, \mu^{(p)}) (|H| z)^d.$$

Proof. This follows from (6) and Proposition 3.34. □

We now turn to the wreath products $H \wr A_n$ and the Weyl groups W_n. Given the correspondence between $\Phi_G(z)$ and $\Phi_P(z)$ described above, it is straightforward to adjust the proof of Proposition 3.17 to obtain a proof of the generalization below, which applies to an arbitrary finite abelian group.

Definition 3.36. *Let $\lambda^{(\pi)}$ be a π-partition and $G = G(\lambda^{(\pi)})$ a finite abelian group of type $\lambda^{(\pi)}$. For $T \leq G$ with $|T| = 2$, define*

$$m_2(T) := \max\big\{|C| : C \text{ a cyclic 2-subgroup containing } T\big\}.$$

Proposition 3.37. *Let $G = G(\lambda^{(\pi)})$ be a finite abelian group. Let $\mu = \lambda^{(2)}$. For each i such that $m_i(\mu) \neq 0$, let $T_i \leq G$ be a subgroup of order two with $m_2(T_i) = 2^i$. Let $M_i = T_i^*$ and set $\Theta_i(z) = \Theta_{M_i}^G(z)$ and $\Theta^i(z) = \Theta_G^{M_i}(z)$. For all $i \in \mathbb{N}$ such that $\mu_i = 0$, set $\Theta_i(z) = \Theta^i(z) = 0$. For a given finite group H, write $\mathcal{S}_{G,H}(z)$*

for $\sum_{n\geq 0}\frac{|Hom(G,H\wr S_n)|}{n!}z^n$*, and* $\mathcal{A}_{G,H}(z)$ *for* $\sum_{n\geq 0}\frac{|Hom(G,H\wr A_n)|}{n!}z^n$*. Write* $\mathcal{W}_G(z)$ *for* $\sum_{n\geq 0}\frac{|Hom(G,W_n)|}{n!}z^n$*. For* $a\in\mathbb{N}$ *define*

$$C_a^{(2)}(z) := \sum_{i<a}(2^{\lambda_i'}-2^{\lambda_{i+1}'})\Theta_i(z)+2^{\lambda_a'}\Theta_a(z),$$

and

$$C_{(2)}^a(z) := \sum_{i<a}(2^{\lambda_i'}-2^{\lambda_{i+1}'})\Theta^i(z)+2^{\lambda_a'}\Theta^a(z).$$

Then for any finite group H *we have*

$$\mathcal{A}_{G,H}(z) = \frac{\mathcal{S}_{G,H}(z)}{2^{\mu_1'}}\left[1+\sum_{a\in\mathbb{N}}(2^{\lambda_1'-\lambda_{a+1}'}-2^{\lambda_1'-\lambda_a'})e^{-C_a^{(2)}(z)}\right],$$

and

$$\mathcal{W}_G(z) = \frac{\mathcal{S}_{G,C_2}(z)}{2^{\mu_1'}}\left[1+\sum_{a\in\mathbb{N}}(2^{\lambda_1'-\lambda_{a+1}'}-2^{\lambda_1'-\lambda_a'})e^{-C_{(2)}^a(z)}\right].$$

Thus we can provide results on finite abelian groups analogous to those stated for finite abelian 2-groups by determining the functions $\Theta_i(z)$ and $\Theta^i(z)$ of Proposition 3.37. We begin with the alterating wreath product case. First we must introduce some additional notation.

Definition 3.38. *For a set* π *of primes, define* $\pi_1 := \pi - \{2\}$*. For a partition* λ *and a nonnegative integer* i*, define*

$$\mathcal{X}(\lambda,i) := \big\{\mu\subseteq\lambda : |\mu|+i=|\lambda|\big\},$$

and

$$l(\lambda,i) := \frac{2^{1+n(\lambda)}}{2^{\lambda_1'-\lambda_{i+1}'}-2^{\lambda_1'-\lambda_i'}}.$$

Note that when examining the alternating wreath product case, we may assume that $\lambda^{(2)}$ is not the empty partition, since if $|G|$ is odd, then the image of any homomorphism from G to $H\wr S_n$ lies in $H\wr A_n$. We will provide the generalization of Lemma 3.26 to arbitrary finite abelian groups by determining the functions $s(d,M,G)$ for all M such that $(G:M)=2$, where $s(d,M,G)$ is as in Subsection 3.4.

Theorem 3.39. *Let* $G=G(\lambda^{(\pi)})$ *be a finite abelian group and let* H *be any finite group. Let* $M<G$ *with* $(G:M)=2$ *and* $m_2(M^*)=2^i$*. Let* $d=\prod_{p\in\pi}p^{d_p}$ *be a*

divisor of $|G|$. For $\mu^{(2)} \in \mathcal{P}(\lambda^{(2)}, i, d_2)$, set

$$k(\lambda^{(2)}, \mu^{(2)}) := \frac{\prod_{j \in I(\lambda^{(2)}, \mu^{(2)})} (1 - 2^{-m_j(\lambda^{(2)})})}{2^{n(\mu^{(2)})}}.$$

Write $\mathcal{P}_2$ for $\mathcal{P}(\lambda^{(2)}, i, d_2)$ and $\mathcal{X}_p$ for $\mathcal{X}(\lambda^{(p)}, d_p)$. Then

$$s(d, M, G) = l(\lambda^{(2)}, i) \sum_{\mu^{(2)} \in \mathcal{P}_2} \sum_{p \in \pi_1} \sum_{\mu^{(p)} \in \mathcal{X}_p} k(\lambda^{(2)}, \mu^{(2)}) \prod_{p \in \pi_1} h_p(\lambda^{(p)}, \mu^{(p)}) f(\mu^{(\pi)}, H).$$

Proof. For any $p \in \pi$, let $P_p(G)$ be the Sylow p-subgroup of G. Define $\mathcal{C}(M^*)$ to be the set of all $X \leq G$ with $|X| = d$ such that $M^* \leq X$ and $P_2(X)$ is cyclic. By Lemma 3.20 and Corollary 3.6, we have

$$s(d, M, G) = \sum_{X \in \mathcal{C}(M)} |\mathrm{Hom}(G/X, H)|.$$

Let $\mathcal{Y}(\lambda^{(\pi)})$ be the set of all π-partitions $\mu^{(\pi)} \subseteq_{(\pi)} \lambda^{(\pi)}$ such that $|\mu^{(p)}| + d_p = |\lambda^{(p)}|$ for all $p \in \pi$. Then there exist nonegative integers $c(\mu^{(\pi)})$, which will be determined below, such that

$$\sum_{X \in \mathcal{C}(M)} |\mathrm{Hom}(G/X, H)| = \sum_{\mu^{(\pi)} \in \mathcal{Y}(\lambda^{(\pi)})} c(\mu^{(\pi)}) |\mathrm{Hom}(G(\mu^{(\pi)}), H)|.$$

Since $G/X \cong G(\mu^{(\pi)})$ if and only if $P_p(G)/P_p(X) \cong G(p, \mu^{(p)})$ for all $p \in \pi$ and any $X \leq G$ can be built by independently choosing $P_p(X)$ for each $p \in \pi$, for any $\mu^{(\pi)} \in \mathcal{Y}(\lambda^{(\pi)})$ we have

$$c(\mu^{(\pi)}) = \prod_{p \in \pi} c_p(\mu^{(\pi)}),$$

where by Corollary 3.14, Lemma 3.23 and Proposition 3.24,

$$c_2(\mu^{(\pi)}) = \begin{cases} \frac{k(\lambda^{(2)}, \mu^{(2)})}{l(\lambda^{(2)}, i)}, & \mu^{(2)} \in \mathcal{P}_2 \\ 0, & \text{otherwise,} \end{cases}$$

and by Corollary 3.6 and Proposition 3.18,

$$c_p(\mu^{(\pi)}) = h_p(\lambda^{(p)}, \mu^{(p)})$$

for all $p \in \pi_1$. □

A long but explicit formula for the exponential generating function of $|\mathrm{Hom}(G, H \wr A_n)|$ can now be obtained from Propostions 3.35 and 3.37 and Theorem 3.39.

The generalization to finite abelian groups for the Weyl groups W_n is straightforward, and the proof of the theorem below is similar to that of Theorem 3.39, using Lemma 3.28 rather than Lemmas 3.20 and 3.23 and Corollary 3.24.

Theorem 3.40. *Let $G = G(\lambda^{(\pi)})$ be a finite abelian group. Let $M < G$ with $(G : M) = 2$. Let $d = \prod_{p\in\pi} p^{d_p}$ be a divisor of $|G|$.*

1. *If $2^{d_2} \geq m_2(M^*)$ then $s(d, G, M) = 0$.*
2. *If $2^{d_2} < m_2(M^*)$ then*

$$s(d,G,M) = \frac{|G(2,\lambda^{(2)})|}{2^{d_2+|\lambda^{(2)}-d_2|}} \prod_{p\in\pi_1} \sum_{\substack{\mu^{(p)}\subseteq\lambda^{(p)}\\ |\mu^{(p)}|+d_p=|\lambda^{(p)}|}} h_p(\lambda^{(p)},\mu^{(p)}).$$

An explicit formula for the exponential generating function of $|\mathrm{Hom}(G, W_n)|$ can be obtained using Propositions 3.35 and 3.37 and Theorem 3.40.

4. Asymptotics of the function $|\mathrm{Hom}(G, W_n)|$

In [12, Sect. 4] a complete asymptotic expansion of the function $|\mathrm{Hom}(G, H \wr S_n)|$ is obtained for arbitrary finite groups G and H in terms of $m = |G|, |H|$, and the numbers $s_G^H(d) = \sum_{(G:U)=d} |\mathrm{Hom}(U,H)|$ for $d \mid m$. Its main term yields the asymptotic formula

$$|\mathrm{Hom}(G, H\wr S_n)| \sim K_G^H\,(n|H|)^{(1-1/m)n}\,\exp\left(-\frac{m-1}{m}\,n + \frac{1}{|H|}\sum_{\substack{d\mid m\\ d<m}} \frac{s_G^H(d)}{d}\,(n|H|)^{d/m}\right)$$

with suitable constant K_G^H, and the quotient

$$|\mathrm{Hom}(G, H\wr S_n)| \Big/ \left(K_G^H\,(n|H|)^{(1-1/m)n}\,\exp\left(-\frac{m-1}{m}\,n + \frac{1}{|H|}\sum_{\substack{d\mid m\\ d<m}} \frac{s_G^H(d)}{d}\,(n|H|)^{d/m}\right)\right)$$

is expanded as a Poincaré series in $(n|H|)^{-1/m}$ with coefficients given explicitly in terms of $m, |H|$, and the $s_G^H(d)$; cf. [12, Theorem 4]. Concerning the asymptotic enumeration of W_n representations the following conjecture is made in [12].

Conjecture 4.1. *For each finite group G there exists a constant $A = A(G) > 1$ such that*

$$|Hom(G, W_n)| = \big(1 + s_G(2)\big)^{-1}\,|Hom(G, C_2 \wr S_n)|\,\Big\{1 + \mathcal{O}(A^{-n^{1/|G|}})\Big\}. \tag{28}$$

Validity of (28) implies in particular that the asymptotic expansion provided by [12, Theorem 4] for $|\mathrm{Hom}(G, C_2 \wr S_n)|$ carries over without change to $(1 + s_G(2))\,|\mathrm{Hom}(G, W_n)|$. Here, we shall verify Conjecture 4.1 in the case when G is dihedral or abelian.

4.1. **Dihedral groups.** For a positive integer l let D_l be the dihedral group of order $2l$.

Proposition 4.2. (a) *For every odd number $l > 0$ and every real number ε satisfying $0 < \varepsilon < 2$ we have*

$$|Hom(D_l, W_n)| = \frac{1}{2}\,|Hom(D_l, C_2 \wr S_n)|\left\{1 + \mathcal{O}(A_1^{-\sqrt{n}})\right\}$$

with

$$A_1 = A_1(l, \varepsilon) := \begin{cases} e^{(2-\varepsilon)}/\sqrt{2}, & l = 1 \\ e^{\sqrt{2}}, & l > 1. \end{cases}$$

(b) *For $l \in 2\mathbb{N}$ and $0 < \varepsilon < 4$ we have*

$$|Hom(D_l, W_n)| = \frac{1}{4}\,|Hom(D_l, C_2 \wr S_n)|\left\{1 + \mathcal{O}(A_2^{-n^{1/4}})\right\}$$

with

$$A_2 = A_2(l, \varepsilon) := \begin{cases} e^{(4-\varepsilon)/2^{3/4}}; & l = 2 \\ e^{2^{5/4}}; & l > 2,\ l/2 \equiv 1 \quad (2) \\ e^{2^{1/4}}; & l/2 \equiv 0 \quad (2). \end{cases}$$

We will need the following consequence of [11, Theorem 1].

Lemma 4.3. *Suppose that the polynomial $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu \in \mathbb{R}[z]$ has degree $m \geq 1$ and meets the conditions*

(i) $c_\mu \geq 0$ *for* $1 \leq \mu \leq m$,
(ii) $c_1 \neq 0$,
(iii) $c_\mu = 0$ *for* $m/2 < \mu < m$.

Let $\exp(P(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$. *Then for $n \geq 1$ the exact order of magnitude of the coefficient α_n is given by*

$$\left(\frac{n}{mc_m}\right)^{-n/m} \exp\left(P\left(\left(\frac{n}{mc_m}\right)^{1/m}\right)\right)\Big/\sqrt{n}.$$

Proof of Proposition 4.2. By (18), (21), and the explicit formulae for Θ^1 and Θ^2 given at the end of Subsection 2.3 we have for $l \geq 1$ and $n \geq 0$

$$|\mathrm{Hom}(D_l, W_n)| = \frac{1}{4}\,|\mathrm{Hom}(D_l, C_2 \wr S_n)|\left\{1 + 2Q_n^{(1)} + Q_n^{(2)}\right\},$$

where

$$Q_n^{(1)} := \big\langle z^n, \exp\big(P_{D_l}^{C_2}(z) - 2(\Theta^1(l)+\Theta^2(l))\big)\big\rangle \big/ \big\langle z^n, \exp\big(P_{D_l}^{C_2}(z)\big)\big\rangle,$$
$$Q_n^{(2)} := \big\langle z^n, \exp\big(P_{D_l}^{C_2}(z) - 4\Theta^2(l)\big)\big\rangle \big/ \big\langle z^n, \exp\big(P_{D_l}^{C_2}(z)\big)\big\rangle,$$
$$P_{D_l}^{C_2}(z) = \frac{1}{2}\sum_{d|l} |\mathrm{Hom}(D_{l/d}, C_2)|(2z)^d + \frac{1}{4}\sum_{d|l} \iota_{l/d}(C_2)(2z)^{2d}/d,$$
$$\Theta^1(l) = \frac{1}{2}\sum_{\substack{d|l\\ d\equiv 1(2)}} (2z)^d, \quad \text{and}$$
$$\Theta^2(l) = \begin{cases} \frac{1}{2}\Bigg(\sum\limits_{\substack{d|k\\ d\equiv 1(2)}} (2z)^d + \sum\limits_{\substack{d|k\\ k/d\equiv 0(2)}} (2z)^{2d}\Bigg), & l = 2k\\ 0, & l \text{ odd}.\end{cases}$$

Since $P_{D_l}^{C_2}(z)$ satisfies the hypotheses of Lemma 4.3 we infer in particular that for $n \geq 1$

$$\big\langle z^n, \exp\big(P_{D_l}^{C_2}(z)\big)\big\rangle \gg$$
$$\left(\frac{2n}{2^{2l}}\right)^{-\frac{n}{2l}} \exp\left(\frac{1}{2}\sum_{d|l} |\mathrm{Hom}(D_{l/d}, C_2)|(2n)^{\frac{d}{2l}} + \frac{1}{4}\sum_{d|l} \iota_{l/d}(C_2)(2n)^{d/l}/d\right)\Big/\sqrt{n},$$

with an implied constant which may depend on l. For odd l we have $Q_n^{(2)} = 1$ and

$$P_{D_l}^{C_2}(z) - 2(\Theta^1(l)+\Theta^2(l)) = \frac{1}{4}\sum_{d|l}(2z)^{2d}/d,$$

whereas for even l

$$P_{D_l}^{C_2}(z) - 2(\Theta^1(l)+\Theta^2(l)) = \sum_{\substack{d|l\\ d\equiv 0(2)}} (2z)^d + \frac{1}{4}\sum_{d|l} \iota_{l/d}(C_2)(2z)^{2d}/d,$$
$$P_{D_l}^{C_2}(z) - 4\Theta^2(l) = \sum_{\substack{d|l\\ l/d\equiv 1(2)}} (2z)^d + \frac{1}{4}\sum_{d|l} \iota_{l/d}(C_2)(2z)^{2d}/d.$$

The polynomials occurring in the numerator of $Q_n^{(1)}$ and $Q_n^{(2)}$ while meeting conditions (i) and (iii) fail to satisfy condition (ii) of Lemma 4.3; but this can be remedied by adding an extra term εz and applying Lemma 4.3 to the resulting polynomials. Fixing a real number ε with $0 < \varepsilon < 2$ we find in this way that for $n \geq 1$

$$\begin{aligned} 0 &\leq \Big\langle z^n, \exp\Big(\frac{1}{4}\sum_{d|l}(2z)^{2d}/d\Big)\Big\rangle \\ &\leq \Big\langle z^n, \exp\Big(\varepsilon z + \frac{1}{4}\sum_{d|l}(2z)^{2d}/d\Big)\Big\rangle \\ &\ll \Big(\frac{2n}{2^{2l}}\Big)^{-\frac{n}{2l}} \exp\Big(\frac{\varepsilon}{2}(2n)^{\frac{1}{2l}} + \frac{1}{4}\sum_{d|l}(2n)^{d/l}/d\Big)/\sqrt{n}. \end{aligned}$$

It follows that for odd l

$$|\mathrm{Hom}(D_l, W_n)| = \frac{1}{2}|\mathrm{Hom}(D_l, C_2 \wr S_n)|\big\{1 + Q_n^{(1)}\big\}$$

with

$$Q_n^{(1)} \ll \exp\left(-(1-\varepsilon/2)(2n)^{\frac{1}{2l}} - \sum_{\substack{d|l\\ d>1}}(2n)^{\frac{d}{2l}}\right) \leq A_1^{-\sqrt{n}}$$

and $A_1 = A_1(l,\varepsilon)$ as defined in the proposition. Similarly, we infer from Lemma 4.3 that for l even, $0 < \varepsilon < 4$, and $n \geq 1$

$$\begin{aligned} &\Big\langle z^n, \exp\big(P_{D_l}^{C_2}(z) - 2(\Theta^1(l) + \Theta^2(l))\big)\Big\rangle \\ &\quad \ll \Big(\frac{2n}{2^{2l}}\Big)^{-\frac{n}{2l}} \exp\Big(\frac{\varepsilon}{2}(2n)^{\frac{1}{2l}} + \sum_{\substack{d|l\\ d\equiv 0(2)}}(2n)^{\frac{d}{2l}} + \frac{1}{4}\sum_{d|l}\iota_{l/d}(C_2)(2n)^{d/l}/d\Big)/\sqrt{n} \end{aligned}$$

and

$$\begin{aligned} &\Big\langle z^n, \exp\big(P_{D_l}^{C_2}(z) - 4\Theta^2(l)\big)\Big\rangle \\ &\quad \ll \Big(\frac{2n}{2^{2l}}\Big)^{-\frac{n}{2l}} \exp\Big(\frac{\varepsilon}{2}(2n)^{\frac{1}{2l}} + \sum_{\substack{d|l\\ l/d\equiv 1(2)}}(2n)^{\frac{d}{2l}} + \frac{1}{4}\sum_{d|l}\iota_{l/d}(C_2)(2n)^{d/l}/d\Big)/\sqrt{n}, \end{aligned}$$

and, hence, for $n \geq 1$

$$Q_n^{(1)} \ll \exp\left(-(2-\varepsilon/2)(2n)^{\frac{1}{2l}} - 2\sum_{\substack{d|l\\ d>1\\ d\equiv 1(2)}}(2n)^{\frac{d}{2l}} - \sum_{\substack{d|l\\ d\equiv 0(2)\\ l/d\equiv 0(2)}}(2n)^{\frac{d}{2l}}\right),$$

$$Q_n^{(2)} \ll \exp\left(-(2-\varepsilon/2)(2n)^{\frac{1}{2l}} - 2\sum_{\substack{d|l\\ d>1\\ l/d\equiv 0(2)}}(2n)^{\frac{d}{2l}}\right).$$

It follows from these estimates that, up to constant, $Q_n^{(1)}$ and $Q_n^{(2)}$ are both bounded from above by $A_2^{-n^{1/4}}$ with A_2 as defined in part (b) of the proposition. The proof of Proposition 4.2 is thus complete. □

4.2. **A partition lemma.** Lemma 4.5 below will play a crucial role in the next subsection, in which we analyze the asymptotic behaviour of the function $|\mathrm{Hom}(G, W_n)|$ for a finite abelian group G. The following observation is preliminary.

Lemma 4.4. *Let λ be a partition and ρ a nonnegative integer. Then we have*

$$|\lambda| = |\lambda - \rho| + \sum_{\kappa=1}^{\rho} \lambda'_\kappa. \tag{29}$$

Proof. For $\rho \geq \lambda_1$ both sides of (29) reduce to $|\lambda| = |\lambda'|$. We prove the assertion for $0 \leq \rho < \lambda_1$ by induction on ρ. For $\rho = 0$ both sides of (29) equal $|\lambda|$. Suppose that (29) holds for some ρ with $0 \leq \rho \leq \lambda_1 - 2$. Let $\lambda = (i_1^{a_1}, i_2^{a_2}, \ldots, i_s^{a_s})$ with $i_1 > i_2 > \ldots > i_s > 0$ and $a_1, a_2, \ldots, a_s > 0$, put $i_{s+1} := 0$, and let k be such that $1 \leq k \leq s$ and $i_{k+1} \leq \rho < i_k$. Then $\lambda' = ((\lambda'_{i_s})^{i_s}, (\lambda'_{i_{s-1}})^{i_{s-1}-i_s}, \ldots, (\lambda'_{i_1})^{i_1-i_2})$, $\lambda'_{\rho+1} = \lambda'_{i_k}$, $\lambda - \rho = ((i_1-\rho)^{a_1}, \ldots, (i_k - \rho)^{a_k})$, and $\lambda - (\rho+1) = ((i_1-\rho-1)^{a_1}, \ldots, (i_k - \rho - 1)^{a_k})$. It follows that $|\lambda - \rho| = |\lambda - (\rho+1)| + \lambda'_{i_k}$ and, hence,

$$\begin{aligned} |\lambda| - |\lambda - (\rho+1)| &= |\lambda| - |\lambda - \rho| + \lambda'_{i_k} \\ &= \sum_{\kappa=1}^{\rho} \lambda'_\kappa + \lambda'_{\rho+1} = \sum_{\kappa=1}^{\rho+1} \lambda'_\kappa. \end{aligned}$$

□

Lemma 4.5. *Given a partition λ and an integer ρ with $0 \leq \rho \leq \lambda_1$ there exists a partition μ such that*

(i) $\mu \subseteq \lambda$,
(ii) $|\mu| + \rho = |\lambda|$,
(iii)

$$2^{\mu'_1} \prod_{\kappa \geq 1} 2^{\mu'_{\kappa+1}(\lambda'_\kappa - \mu'_\kappa)} \binom{\lambda'_\kappa - \mu'_{\kappa+1}}{\mu'_\kappa - \mu'_{\kappa+1}}_2 \geq 2^{|\lambda| - |\lambda - \rho| + \lambda'_{\rho+1} - \rho}.$$

Proof. For $\mu \subseteq \lambda$ we have

$$\begin{aligned}\prod_{\kappa\geq 1}\binom{\lambda'_\kappa-\mu'_{\kappa+1}}{\mu'_\kappa-\mu'_{\kappa+1}}_2 &= \prod_{\kappa\geq 1}\prod_{j=0}^{\mu'_\kappa-\mu'_{\kappa+1}-1}\frac{2^{\lambda'_\kappa-\mu'_{\kappa+1}-j}-1}{2^{\mu'_\kappa-\mu'_{\kappa+1}-j}-1}\\ &\geq \prod_{\kappa\geq 1}\prod_{j=0}^{\mu'_\kappa-\mu'_{\kappa+1}-1} 2^{\lambda'_\kappa-\mu'_\kappa}\\ &= 2^{\langle\mu',\lambda'\rangle-||\mu'||^2-\sum\mu'_{\kappa+1}(\lambda'_\kappa-\mu'_\kappa)},\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ and $||\cdot||$ denote the inner product and the norm, respectively, in the Hilbert space $l^2(\mathbb{R})$. From this estimate and Lemma 4.4 we conclude that the inequality

$$\mu'_1+\langle\mu',\lambda'\rangle+\rho\geq ||\mu'||^2+\sum_{\kappa=1}^{\rho+1}\lambda'_\kappa \tag{30}$$

is a sufficient condition for a partition $\mu\subseteq\lambda$ to satisfy (iii). Let $\lambda'=(i_1^{a_1},i_2^{a_2},\dots,i_s^{a_s})$ with $i_1>i_2>\dots>i_s>0$ and $a_1,a_2,\dots,a_s>0$, and for $0\leq j\leq s$ put $b_j:=\sum_{\kappa=1}^{j}a_\kappa$. Since $b_0=0$ and $b_s=\lambda_1$, there exists an integer k with $0<k\leq s$ and $b_{k-1}\leq\rho\leq b_k$. Define

$$\mu'_\kappa:=\begin{cases}\lambda'_\kappa-1, & 1\leq\kappa\leq b_{k-1}\ \ or\ \ b_{k-1}+b_k-\rho<\kappa\leq b_k\\ \lambda'_\kappa, & all\ \ other\ \ \kappa\in\mathbb{N}.\end{cases}$$

Clearly, $\mu'=(\mu'_1,\mu'_2,\dots)$ is a partition such that $\mu'\subseteq\lambda'$ and $|\mu'|+\rho=|\lambda'|$. An immediate calculation gives

$$\langle\mu',\lambda'\rangle=||\lambda'||^2-\sum_{\kappa=1}^{b_{k-1}}\lambda'_\kappa-\sum_{\kappa=b_{k-1}+b_k-\rho+1}^{b_k}\lambda'_\kappa$$

and

$$||\mu'||^2=||\lambda'||^2-2\sum_{\kappa=1}^{b_{k-1}}\lambda'_\kappa-2\sum_{\kappa=b_{k-1}+b_k-\rho+1}^{b_k}\lambda'_\kappa\ \ +\rho.$$

It follows that μ' satisfies (30) if and only if

$$\mu'_1\geq\lambda'_{\rho+1}. \tag{31}$$

We have

$$\mu'_1=\begin{cases}\lambda'_1, & \rho<b_1\\ \lambda'_1-1, & otherwise.\end{cases}$$

Hence, if $\rho<b_1$ then both sides of (31) equal i_1. If on the other hand $\rho\geq b_1$ then (31) is equivalent to the assertion that $i_1>\lambda'_{\rho+1}$, which holds since in this case $\lambda'_{\rho+1} \leq i_2$. We conclude that the partition μ (the conjugate of μ') satisfies (i) – (iii). □

4.3. **Abelian groups.** Let $\lambda^{(\pi)}$ be a π-partition and let $G = G(\lambda^{(\pi)})$ be a finite abelian group of type $\lambda^{(\pi)}$. We shall assume that $|\lambda^{(2)}| > 0$, since otherwise (28) holds trivially for G. Put $\lambda := \lambda^{(2)}$, $l := \lambda_{\lambda'_1}$, and $m' := m/2^{|\lambda|}$, where $m = \prod_{p\in\pi} p^{|\lambda^{(p)}|}$ is the order of G.

Proposition 4.6. *For G as above and every real number ε satisfying $0 < \varepsilon < 2^{\lambda'_1}$ we have*

$$|Hom(G, W_n)| = \frac{|Hom(G, C_2 \wr S_n)|}{1 + s_G(2)} \Big\{1 + \mathcal{O}\big(A^{-n^{2^{l-|\lambda|-1}}}\big)\Big\}$$

with

$$A = A(\lambda^{(\pi)}, \varepsilon) := \begin{cases} e^{(2^{\lambda'_1}-\varepsilon)/2^{\frac{m-1}{m}}}, & m' = 1 \text{ and } m_1(\lambda) \neq 0 \\ e^{2^{\lambda'_l - |\lambda-(l-1)|-l}/m'}, & \text{otherwise.} \end{cases}$$

Proof. By Propositions 3.17 and 3.35 and Theorem 3.38 we have

$$|Hom(G, W_n)| = \frac{|Hom(G, C_2 \wr S_n)|}{1 + s_G(2)} \Big\{1 + \sum_{a\in\mathbb{N}} \big(2^{\lambda'_1-\lambda'_{a+1}} - 2^{\lambda'_1-\lambda'_a}\big)\, Q_n^{(a)}\Big\} \tag{32}$$

where

$$Q_n^{(a)} := \big\langle z^n, \exp\big(P_G^{C_2}(z) - C^a(z)\big)\big\rangle \big/ \big\langle z^n, \exp\big(P_G^{C_2}(z)\big)\big\rangle,$$

$$\begin{aligned} P_G^{C_2}(z) &= \frac{1}{2}\sum_{d|m}\frac{1}{d} \sum_{\substack{\mu^{(\pi)}\subseteq_{(\pi)}\lambda^{(\pi)} \\ j(\lambda^{(\pi)},\mu^{(\pi)})=d}} 2^{(\mu^{(2)})'_1} \prod_{p\in\pi} h_p(\lambda^{(p)}, \mu^{(p)})\ (2z)^d \\ &= \sum_{\rho=0}^{|\lambda|}\sum_{d'|m'} \frac{2^{2^\rho d'-\rho-1}}{d'} \Bigg[\prod_{p\in\pi_1} \sum_{\substack{\mu^{(p)}\subseteq\lambda^{(p)} \\ |\mu^{(p)}|+d'_p=|\lambda^{(p)}|}} h_p(\lambda^{(p)}, \mu^{(p)})\Bigg] \Bigg[\sum_{\substack{\mu\subseteq\lambda \\ |\mu|+\rho=|\lambda|}} 2^{\mu'_1}\, h_2(\lambda, \mu)\Bigg] z^{2^\rho d'} \end{aligned}$$

with $d' = \prod_{p\in\pi_1} p^{d'_p}$, and, for $a \in \mathbb{N}$ with $m_a(\lambda) \neq 0$,

$$C^a(z) = \sum_{\rho=0}^{a-1}\sum_{d'|m'} \frac{2^{2^\rho d'+|\lambda|-|\lambda-\rho|+\lambda'_{\rho+1}-2\rho-1}}{d'} \Bigg[\prod_{p\in\pi_1} \sum_{\substack{\mu^{(p)}\subseteq\lambda^{(p)} \\ |\mu^{(p)}|+d'_p=|\lambda^{(p)}|}} h_p(\lambda^{(p)}, \mu^{(p)})\Bigg] z^{2^\rho d'}.$$

Since the polynomial $P_G^{C_2}(z)$ satisfies the hypotheses of Lemma 4.3 we infer in particular that

$$\big\langle z^n, \exp\big(P_G^{C_2}(z)\big)\big\rangle \gg \Big(\frac{2n}{2^m}\Big)^{-n/m} \exp\big(P_G^{C_2}\big((2n)^{1/m}/2\big)\big)/\sqrt{n}, \quad n\geq 1. \tag{33}$$

Moreover, since

$$\deg(C^a(z)) \leq 2^{\lambda_1 - 1} m' \leq m/2, \quad m_a(\lambda) \neq 0$$

the polynomials $P_a(z) := P_G^{C_2}(z) - C^a(z)$ occurring with nonzero coefficient in (32) satisfy the gap condition (iii) of Lemma 4.3 and have the same leading term as $P_G^{C_2}(z)$. On the other hand

$$\langle z, P_G^{C_2}(z)\rangle = 2^{\lambda_1'} = \langle z, C^a(z)\rangle, \quad m_a(\lambda)\neq 0,$$

i.e., none of the $P_a(z)$ satisfies condition (ii) of Lemma 4.3. By Lemma 4.5 the polynomial $P_{\lambda_1}(z)$ has nonnegative coefficients, implying that all polynomials $P_a(z)$ occurring in (32) have this property. Applying Lemma 4.3 we find that for every $\varepsilon > 0$

$$\big\langle z^n, \exp\big(P_a(z)\big)\big\rangle \ll \Big(\frac{2n}{2^m}\Big)^{-n/m} \exp\Big(P_a\big((2n)^{1/m}/2\big) + \varepsilon (2n)^{1/m}/2\Big)/\sqrt{n} \qquad (n\geq 1,\ m_a(\lambda)\neq 0). \tag{34}$$

From (33) and (34) it follows that for $n \geq 1$, $m_a(\lambda) \neq 0$, and every ε with $0 < \varepsilon < 2^{\lambda_1'}$

$$Q_n^{(a)} \ll \exp\Bigg(- \frac{2^{\lambda_1'} - \varepsilon}{2}(2n)^{1/m} - \sum_{\substack{\rho=0 \\ (\rho,d')\neq(0,1)}}^{a-1} \sum_{d'|m'} \frac{2^{2^\rho d' + |\lambda| - |\lambda-\rho| + \lambda'_{\rho+1} - 2\rho - 1}}{d'} \Bigg[\prod_{p\in\pi_1} \sum_{\substack{\mu^{(p)} \subseteq \lambda^{(p)} \\ |\mu^{(p)}| + d'_p = |\lambda^{(p)}|}} h_p(\lambda^{(p)}, \mu^{(p)}) \Bigg] \big((2n)^{1/m}/2\big)^{2^\rho d'} \Bigg). \tag{35}$$

If $(a, m') = (1,1)$, the right-hand side of (35) equals

$$\big(\mathrm{e}^{(2^{\lambda'_1} - \varepsilon)/2^{\frac{m-1}{m}}}\big)^{-n^{1/m}},$$

while for $(a, m') \neq (1,1)$ it is bounded from above by

$$\exp\Big(- \frac{2^{\lambda'_a - |\lambda - (a-1)| - a}}{m'} n^{2^{a - |\lambda| - 1}}\Big).$$

Since the order of magnitude of the latter term decreases with increasing a the assertions of Proposition 4.6 follow. $\square$

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