Math 131E | Krantz
Fall, 2021 | December 9, 2021

Second Midterm Exam

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear. This exam has 10 questions and is worth 100 points. You will have one hour to take this exam.

If you need extra space then write on the back of the page.

(10 points) **1.** Calculate the definite integral

$$\int_2^4 \frac{\ln x}{x}\, dx\,.$$

Let $u = \ln x$. Then $\frac{du}{dx} = \frac{1}{x}$ so $du = \frac{1}{x}dx$.

Therefore $\int_2^4 \frac{\ln x}{x}\, dx = \int_2^4 (\ln x) \cdot \frac{1}{x}\, dx = \int_{\ln 2}^{\ln 4} u\, du$

$$= \frac{u^2}{2}\Big]_{\ln 2}^{\ln 4} = \frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2}.$$

(10 points) **2.** A spherical balloon is being inflated at the rate of 4 cubic inches per minute. At what rate is the radius changing when the radius is equal to 6?

$$V = \frac{4}{3}\pi r^3. \text{ So } \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

So $$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi \cdot 6^2} \cdot 4 = \frac{1}{36\pi}.$$

(10 points) **3.** Calculate the indefinite integral

$$\int (\sin x)^2 \cdot \cos x \, dx\,.$$

Let $u = \sin x$. So $\frac{du}{dx} = \cos x$ hence $du = \cos x \, dx$.

As a result,

$$\int (\sin x)^2 \cdot \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C.$$

$$= \frac{(\sin x)^3}{3} + C.$$

(10 points) **4.** The position of a particle moving along a straight line is give by $s(t) = 2t^2 - 8t + 4$. What is the total distance that the particle travels from $t = 1$ to $t = 6$? [*Hint:* Remember that the particle travels forward some of the time and backward some of the time. You must take this fact into account.]

$v(t) = s'(t) = 4t - 8$. We see that $v' < 0$ for $1 < t < 2$ and $v' > 0$ for $2 < t < 6$.

$\int_1^2 v'(t)\,dt = \int_1^2 4t - 8\,dt = 2t^2 - 8t\Big]_1^2 = (8-16) - (2-8) = -2.$

$\int_2^6 v'(t)\,dt = \int_2^6 4t - 8\,dt = 2t^2 - 8t\Big]_2^6 = (72-48) - (8-16)$
$= 32.$

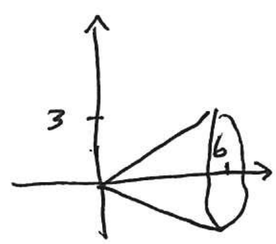
Total distance traveled is $-(-2) + 32 = 34$.

(10 points) **5.** Calculate the volume of the cone with base having radius 3 inches and height 6 inches.

$V = \int_0^6 \pi \left(\frac{1}{2}x\right)^2 dx = \int_0^6 \frac{\pi}{4} x^2\,dx$

$= \frac{\pi}{4} \cdot \frac{x^3}{3}\Big]_0^6 = \frac{\pi}{4} \cdot \frac{216}{3}$

$= \frac{\pi}{4} \cdot 72 = 18\pi.$



(10 points) **6.** Given that

$$xy + y^3 = 2xy\,,$$

use implicit differentiation to calculate dy/dx at the point $(1,1)$.

$$y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$$\frac{dy}{dx}(x + 3y^2 - 2x) = y$$

$$\frac{dy}{dx} = \frac{y}{x+3y^2-2x}.$$

At the point $(1,1)$,

$$\frac{dy}{dx} = \frac{1}{1+3-2} = \frac{1}{2}.$$

(10 points) **7.** Use calculus to sketch the graph of

$$y = x^3 - 3x^2 - 9x + 10\ .$$

In particular, show where the graph is increasing and decreasing and show where it is concave up and concave down.

$y' = 3x^2 - 6x - 9$

$= 3(x^2 - 2x - 3) = 3(x-3)(x+1).$

So $3, -1$ are critical points

$y'' = 6x - 6.$

Hence 3 is a local minimum,

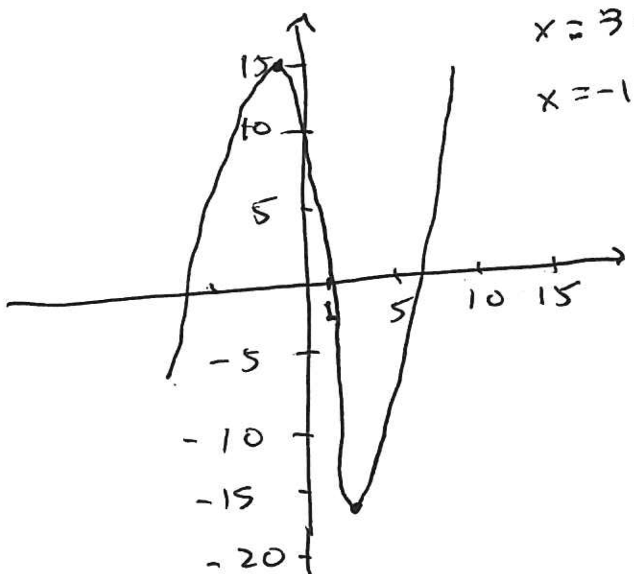
-1 is a local maximum.

The curve is concave up for $x > 1$.

The curve is concave down for $x < 1$.

$x = 3 \Leftrightarrow y = -17$

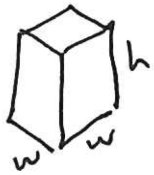
$x = -1 \Leftrightarrow y = 15$



concave down

concave up

(10 points) **8.** A rectangular box with square base and without a top is made from material that costs 10 cents per square inch for the bottom and 5 cents per square inch for the sides. The box is to hold 100 cubic inches. What dimensions will give the cheapest box?



$$100 = w^2 h \Rightarrow h = \frac{100}{w^2}.$$

$$C = 10 \cdot w^2 + 4 \cdot 5 \cdot wh$$

$$= 10 w^2 + 20 \cdot w \cdot \frac{100}{w^2} = 10 w^2 + \frac{2000}{w}.$$

$$c' = 20 w - \frac{2000}{w^2} = 0 \Rightarrow 20 w^3 = 2000$$

$$w^3 = 100$$

$$w = 100^{1/3}$$

$$h = \frac{100}{w^2} = 100^{1/3}$$

(10 points) **9.** What is the area between the curves $y = x^2 - 2$ and $y = -x^2 + 4$?

$$x^2 - 2 = -x^2 + 4$$

$$2x^2 = 6$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} (-x^2+4) - (x^2-2)\, dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} 6 - 2x^2\, dx = 6x - \frac{2}{3} x^3 \Big]_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \left(6\sqrt{3} - \frac{2}{3} 3^{3/2}\right) - \left(-6\sqrt{3} + \frac{2}{3} 3^{3/2}\right)$$

$$= 12\sqrt{3} - \frac{4}{3} 3^{3/2}$$

$$= \sqrt{3}\left(12 - \frac{4}{3} \cdot 3\right) = 8\sqrt{3}.$$

(10 points) **10.** Strontium 90 has a half life of 30 days. A sample has mass 6 grams. Find the mass remaining after 50 days.

$$S'(t) = k \cdot S(t)$$

$$S(t) = Ce^{kt}.$$

$$S(0) = 6 \Rightarrow 6 = C$$

$$S(30) = 3 \Rightarrow 3 = 6e^{k \cdot 30}$$

$$e^{k \cdot 30} = \frac{1}{2}$$

$$k \cdot 30 = \ln \frac{1}{2} = -\ln 2$$

$$k = -\frac{1}{30} \ln 2$$

$$\therefore \; S(t) = 6 \cdot e^{-\frac{1}{30}(\ln 2) \cdot t}$$

$$S(50) = 6 \cdot e^{-\frac{1}{30}(\ln 2) \cdot 50}$$

$$= 6 \cdot e^{-\frac{5}{3} \cdot \ln 2}.$$