New Results on the Bergman Kernel of the Worm Domain in Complex Space

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1. Introduction

In 1977, Diederich and Fornæss [DIF] constructed a counterexample to the long-standing conjecture that (the closure of) a smoothly bounded, pseudoconvex domain in complex space is the decreasing intersection of smooth, pseudoconvex domains. The smoothly bounded domain Ω_β that they constructed has become known as the "worm". The worm is smoothly bounded and pseudoconvex. In fact it is a counterexample to several important questions:

- The worm Ω_β is *not* the decreasing intersection of smooth, pseudoconvex domains.
- There is a function f that is C^∞ on $\overline{\Omega}_\beta$, holomorphic on Ω_β, and such that f cannot be approximated uniformly on $\overline{\Omega}_\beta$ by functions holomorphic on a neighborhood of $\overline{\Omega}_\beta$.
- The domain Ω_β does *not* have a global plurisubharmonic defining function.

The general concept of the worm domain has several concrete realizations. Two of these that will be important for us are:

(i) The unbounded, non-smooth worm

$$D_\beta \equiv \Big\{(\zeta_1,\zeta_2)\in\mathbb{C}^2 : \mathrm{Re}\,(\zeta_1 e^{-i\log|\zeta_2|^2}) > 0\,, \big|\log|\zeta_2|^2\big| < \beta - \frac{\pi}{2}\Big\}$$

(ii) The bounded, smooth worm

$$\Omega_\beta = \{(z_1,z_2) : |z_1 + e^{i\log|z_2|^2}|^2 < 1 - \eta(\log|z_2|^2)\}\,,$$

where

(a) $\eta \geq 0$, η is even, η is convex;
(b) $\eta^{-1}(0) = I_{\beta-\pi/2} \equiv [-\beta + \frac{\pi}{2}, \beta - \frac{\pi}{2}]$;
(c) There exists a number $a > 0$ such that $\eta(x) > 1$ if $x < -a$ or $x > a$;
(d) $\eta'(x) \neq 0$ if $\eta(x) = 1$.

It should be stressed that these two renditions of the worm, D_β and Ω_β, are *not* biholomorphically equivalent. But there are devices (see [BAR3]) for passing back and forth between the two domains. The essential feature of a worm domain is that the Euclidean normal vector to the boundary changes (rotates) rather rapidly as $|z_2|$ ranges over its values. In fact all of the important examples based on the worm mandate that β be sufficiently large, which simply means that the normal rotates a sufficient number of times.

Let $U \subseteq \mathbb{C}^n$ be a bounded domain (i.e., a connected, open set). Let $L^2(U)$ be the usual space of square integrable, measurable functions on U (with respect to ordinary Lebesgue volume measure). Let $A^2(U) \subseteq L^2(U)$ be the square-integrable holomorphic functions. Of course $A^2(U)$ is a Hilbert subspace of $L^2(U)$.

Let

$$P : L^2(U) \to A^2(U)$$

be the Hilbert space projection. Then P is representable by an integral formula:

$$P_U f(z) = \int_U f(\zeta) K_U(z,\zeta)\, dV(\zeta)\,.$$

The kernel $K_U(z,\zeta)$ is called the *Bergman kernel* for U (see [KRA1] for details on the Bergman kernel and related ideas).

If U is a smoothly bounded domain then certainly $C^\infty(\overline{U})$ is a dense subspace of $L^2(U)$. In the paper [BEL1], Steven Bell introduced the following important paradigm: If the Bergman projection maps $C^\infty(\overline{U})$ into $C^\infty(\overline{U})$ then U is said to satisfy *Condition R*. The most important fact about Condition R is this:

> **Theorem: [Bell, 1981]** Let $U_1, U_2 \subseteq \mathbb{C}^n$ be smoothly bounded, pseudoconvex domains that satisfy Condition R. Then any biholomorphic mapping $\varphi : U_1 \to U_2$ will extend to a diffeomorphism of $\overline{U}_1$ to $\overline{U}_2$.

This result generalizes the foundational theorem of Fefferman [FEF], and extends the ideas to a broader class of domains. The boundary behavior of biholomorphic and proper mappings has been a subject of intense study for the past 30 years. Condition R has thus proved to be a powerful and central tool in the function theory of several complex variables.

Standard methods for establishing Condition R on a given domain are

(a) The $\overline{\partial}$-Neumann problem [KRA2], [CSH];
(b) Symmetries [BOS];
(c) Control of the normals to the boundary [BAR1].

It is conjectured that *any* biholomorphic mapping of smoothly bounded, pseudoconvex domains (and, more generally, *all* smoothly bounded domains) will extend to diffeomorphisms of the closures. [This problem is called the *Mapping Conjecture*.] Thus it was natural to suspect that Condition R will hold on any smoothly bounded, pseudoconvex domain.

It came as something of a surprise when, in 1984, David Barrett [BAR] produced a smoothly bounded (non-pseudoconvex) domain in $\mathbb{C}^2$ on which Condition R fails. In 1991, Kiselman showed that the Domain D_β (which is certainly pseudoconvex) fails to satisfy Condition R. While Kiselman's arguments are elegant and compelling, his result is not entirely satisfactory because the domain D_β does *not* have smooth boundary.

In 1992, Barrett [BAR3] built on Kiselman's ideas and showed that the *smooth* worm Ω_β has the property that the Bergman projection P does not map W^s to W^s, where W^s is the s-order Sobolev space and s is sufficiently large. Barrett does his analysis on the non-smooth worm, but then uses an exhaustion procedure to get a negative result on the smooth worm.

All of this is a prelude to the theorem of Christ [CHR]. He showed that the *smooth* worm Ω_β does not satisfy Condition R.

Christ's result is definitive, and shows that we must seek tools other than Condition R if we are to resolve the Mapping Conjecture. Christ's techniques are deep and difficult. It is worthwhile to have other methods for exploring this central collection of ideas.

It should be stressed that the work of Kiselman, Barrett, and Christ uses Bergman theory (as it must). But these works decompose the Bergman space into infinitely many invariant subspaces, and actually (because they are in pursuit of a *negative result*) calculate the kernel and perform the necessary estimates on *just one* of these subspaces—namely the space $\mathcal{H}^{-1}$ (see below for the definition). This particular choice is dictated by the fact that, when the index equals -1, a number of complicated terms cancel out and the calculation is thus simplified. As a result, until now, nothing explicit has been known about the full Bergman kernel or full Bergman projection on the worm.

2. The Bergman Kernel for the Worm

In the work that is being announced here, we actually calculate the Bergman kernel for the non-smooth worm D_β. More precisely, we can write an asymptotic expansion for the kernel in this form:

$$K(\zeta,\omega) = \mathcal{K}(\zeta,\omega) + \mathcal{E}(\zeta,\omega)\,.$$

Here $\mathcal{K}$ is the "principal term" of the expansion for the kernel, and $\mathcal{E}$ is the error. The term $\mathcal{E}$ is bounded, and all of its derivatives are bounded. So mapping properties of the kernel and other properties of the kernel and projection may be determined simply by studying $\mathcal{K}$. The full asymptotic for the Bergman kernel on D_β is as follows:

Theorem: Let c_0 be a positive fixed constant. Let χ_1 be a smooth cuf-off function on the real line, supported on $\{x : |x| \leq 2c_0\}$, identically 1 for $|x| < c_0$. Set $\chi_2 = 1 - \chi_1$.

Let $\beta > \pi$ and let $\nu_\beta = \pi/(2\beta - \pi)$. Let h be fixed, with

$$\nu_\beta < h < \min(1, 2\nu_\beta)\,. \tag{1}$$

There exist functions $G_1, G_2, \ldots, G_9$ and $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_9$, holomorphic in ζ and anti-holomorphic in ω, for $\zeta = (\zeta_1, \zeta_2)$, $\omega = (\omega_1, \omega_2)$ varying in $\overline{D'_\beta} \setminus \{0\}$, such that

$$\partial^\alpha_\zeta \partial^\gamma_{\overline{\omega}} G(\zeta,\omega) = \mathcal{O}\big(|\zeta_1|^{-|\alpha|}|\omega_1|^{-|\gamma|}\big) \qquad \text{as} \quad |\zeta_1|, |\omega_1| \to 0\ ,$$

where G denotes any of the functions G_j, $\tilde{G}_j$, and such that the following holds. Set

$$
\begin{aligned}
H_b(\zeta,w) = & \frac{G_1(\zeta,w)}{(i\log(\zeta_1/\overline{\omega}_1)+2\beta)^2(e^{\tilde{\beta}}-\zeta_2\overline{\omega}_2)^2} \\
& \quad + \frac{G_2(\zeta,w)}{(i\log(\zeta_1/\overline{\omega}_1)+2\beta)^2\big((\zeta_1/\overline{\omega}_1)^{-i/2}e^{-\pi/2}-\zeta_2\overline{\omega}_2\big)^2} \\
& + \frac{G_3(\zeta,w)}{\big((\zeta_1/\overline{\omega}_1)^{-i/2}e^{\pi/2}-\zeta_2\overline{\omega}_2\big)^2(e^{\tilde{\beta}}-\zeta_2\overline{\omega}_2)^2} \\
& \quad + \frac{G_4(\zeta,w)}{(i\log(\zeta_1/\overline{\omega}_1)-2\beta)^2\big((\zeta_1/\overline{\omega}_1)^{-i/2}e^{\pi/2}-\zeta_2\overline{\omega}_2\big)^2} \\
& \quad + \frac{G_5(\zeta,w)}{(i\log(\zeta_1/\overline{\omega}_1)-2\beta)^2(e^{-\tilde{\beta}}-\zeta_2\overline{\omega}_2)^2} \\
& + \frac{G_6(\zeta,w)}{\big((\zeta_1/\overline{\omega}_1)^{-i/2}e^{-\pi/2}-\zeta_2\overline{\omega}_2\big)^2(e^{-\tilde{\beta}}-\zeta_2\overline{\omega}_2)^2} \\
& + \frac{G_7(\zeta,w)}{(i\log(\zeta_1/\overline{\omega}_1)+2\beta)^2(e^{\tilde{\beta}}-\zeta_2\overline{\omega}_2)\big((\zeta_1/\overline{\omega}_1)^{-i/2}e^{-\pi/2}-\zeta_2\overline{\omega}_2\big)} \\
& + \frac{G_8(\zeta,w)}{(i\log(\zeta_1/\overline{\omega}_1)-2\beta)^2(e^{-\tilde{\beta}}-\zeta_2\overline{\omega}_2)\big((\zeta_1/\overline{\omega}_1)^{-i/2}e^{\pi/2}-\zeta_2\overline{\omega}_2\big)} + G_9(\zeta,w) \\
\equiv\ & H_1(\zeta,\omega)+\cdots+H_8(\zeta,\omega)+G_9(\zeta,\omega)\ . \qquad (2)
\end{aligned}
$$

Define $H_{\tilde{b}}$ by replacing $G_1,\ldots,G_9$ by $\tilde{G}_1,\ldots,\tilde{G}_9$ and $H_1,\ldots,H_8$ by $\tilde{H}_1,\ldots,\tilde{H}_8$ respectively.

Then, setting $t=|\zeta_1|-|\omega_1|$, we have this asymptotic expansion for the Bergman kernel on D_β:

$$
\begin{aligned}
& K_{D_\beta}\big((\zeta_1,\zeta_2),(\omega_1,\omega_2)\big) \\
& = \chi_1(t)\frac{H_b(\zeta,\omega)}{\zeta_1\overline{\omega}_1} + \chi_2(t)\Bigg\{\bigg(\frac{|\zeta_1|}{|\omega_1|}\bigg)^{-h\mathrm{sgn}t} e^{-h\mathrm{sgn}t\cdot(\arg\zeta_1+\arg\omega_1)}\frac{H_{\tilde{b}}(\zeta,\omega)}{\zeta_1\overline{\omega}_1} \\
& \quad + \bigg(\frac{|\zeta_1|}{|\omega_1|}\bigg)^{-\nu_\beta\mathrm{sgn}t} e^{-\nu_b\mathrm{sgn}t\cdot(\arg\zeta_1+\arg\omega_1)}\bigg(\frac{g_1(\zeta_1,\omega_1)}{\zeta_1\overline{\omega}_1}\cdot\frac{1}{\big((\zeta_1/\overline{\omega_1})^{-i/2}e^{\pi/2}-z_2\overline{\omega}_2\big)^2} \\
& \qquad + \frac{g_2(\zeta,\omega)}{\zeta_1\overline{\omega}_1}\cdot\frac{1}{\big((\zeta_1/\overline{\omega}_1)^{-i/2}e^{-\pi/2}-z_2\overline{\omega}_2\big)^2}\bigg)\Bigg\}\,.
\end{aligned}
$$

One can see immediately that, when β is sufficiently large ($\beta > 3\pi/2$ will do), then the term $\left(\frac{\zeta_1}{\overline{\omega}_1}\right)^{\nu_\beta}$ causes trouble: the Bergman projection cannot map $C^\infty(\overline{\Omega})$ to $C^\infty(\overline{\Omega})$. In addition, one recovers Ligocka's result [LIG]: that the Bergman kernel on the worm domain is *not* smooth on $\overline{W} \times \overline{W} \setminus$ (diagonal).

One useful consequence of this nice, explicit form for the Bergman kernel is that we can calculate the mapping properties of the Bergman projection on $L^p(\Omega)$. In fact we have:

Theorem: The Bergman projection

$$P : L^2(D_\beta) \to A^2(D_\beta)$$

is bounded on $L^p(D_\beta)$ for

$$\frac{2}{1+\nu_\beta} < p < \frac{2}{1-\nu_\beta}$$

and unbounded on $L^p(D_\beta)$ for either

$$p \leq \frac{2}{1+\nu_\beta} \quad \text{or} \quad p \geq \frac{2}{1-\nu_\beta}$$

as long as $\beta > 3\pi/2$.

It is noteworthy that, as $\beta \to +\infty$, the parameter $\nu_\beta \to 0$ and hence the range of p for which the Bergman projection is bounded on L^p shrinks to the singleton $\{2\}$.

We may mention that there is another useful presentation of the nonsmooth worm D_β, called D'_β, in which the slices are planar strips rather than halfplanes. One may obtain an asymptotic expansion for the Bergman kernel of D'_β by a simple application of the standard transformation formula for Bergman kernels (see [KRA1]). A curious feature, however, is that the Bergman projection on D'_β is bounded on L^p for $1 < p < \infty$.

3. Some Remarks about the Proof

The calculation of the Bergman kernel for the worm D_β is long and difficult. The idea is, emulating Kiselman and Barrett, to define $\mathcal{H}^j(D_\beta)$ to consist of those elements f of $\mathcal{H}(D_\beta) \equiv A^2(D_\beta)$ that satisfy $f(z_1, e^{i\theta}z_2) = e^{ij\theta}f(z_1,z_2)$. Elementary Fourier analysis shows that $\mathcal{H} = \bigoplus \mathcal{H}^j$. Then we use Hilbert space theory and Fourier analysis to obtain a formula for the Bergman kernel K_j of each $\mathcal{H}^j$. In fact it is given by

$$K_j(\omega,\zeta) = k_j(\omega_1,\zeta_1)\omega_2^j\overline{\zeta}_2^j\,.$$

Here

$$k_j(\omega_1,\zeta_1) = \left[\frac{e^{i\zeta\xi}}{\widehat{\lambda}_j(-2i\xi)}\right]^\vee$$

(where $^\vee$ is the inverse Fourier transform),

$$\lambda_j(s) = \chi_{\pi/2} * [e^{(j+1)(\cdot)}\chi_{\beta-\pi/2}] \, ,$$

and χ_s is the characteristic function of the interval $[-s, s]$ when $s \geq 0$.

Of course the full Bergman kernel for the worm is just the sum of the kernels K_j for the component subspaces. The principal term for the Bergman kernel is separated from the error term when we approximate the sinh (which obviously arises in the Fourier transform calculations) by an exponential.

The verification of the failure of Condition R, and of Ligocka's theorem, follows from inspection of the formula for the Bergman kernel.

The necessary estimates for the result on mappings of L^p spaces arise from an application of a refined form of Schur's lemma.

4. Further Explorations

It is hoped that techniques of Christ [CHR], [SIU] and others may be adapted to obtain results on the smooth worm. The analysis will be more difficult on that domain because its boundary no longer resembles a product.

It should be noted that So-Chin Chen [CHE] has actually calculated all the biholomorphic self-maps of the worm Ω_β. These maps may be written down explicitly, and they plainly extend smoothly to the boundary. One might anticipate that a more complete understanding of the Bergman kernel for the smooth worm Ω_β will allow us to analyze biholomorphic mappings of the worm to other smoothly bounded, pseudoconvex domains.

We plan to develop all these ideas in future papers.

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