The Hartogs Extension Phenomenon Redux

by Steven G. Krantz[1][2]

American Institute of Mathematics, Palo Alto, California

Abstract: *We examine the classical Hartogs extension phenomenon from a number of different points of view. Several different proofs are offered, some of them new. Connections with other parts of the subject are developed.*

Keywords: Hartogs phenomenon; pseudoconvexity; holomorphic

AMS Subject Classifications: 32E05, 32E40, 32E99

0 Introduction

The Hartogs extension phenomenon was discovered by Fritz Hartogs in 1906. Along with Poincaré's result about the biholomorphic inequivalence of the ball and the bidisc, this theorem helped to establish the independent essence of the function theory of several complex variables. This was not a trite extension of the one-variable theory. It is in fact an entirely new enterprise.

At about the same time that Hartogs presented his original proof on the polydisc, Poincaré developed an alternative proof on the ball—using spherical harmonics! We shall present a spherical harmonics proof, but *not* Poincaré's, below.

The original approach to the Hartogs extension phenomenon begged important topological questions. One wants to develop a version of this result on virtually *any* domain (at least one with the right topology). But there are nasty analytic continuation questions that were a roadblock for about 100 years. No less an eminence than William Fogg Osgood [OSG] published a classical proof that addressed these questions, but his proof is considered to have been flawed. It is only recently that Merker and Porten in [MEP] have been able to rigorously carry out Osgood's program and produce a classical topological proof. We shall discuss these issues in what follows.

---

[1]Email: skrantz@aimath.org.

[2]The author would like to thank the American Institute of Mathematics for its hospitality and support during this work.

1 Preliminaries

We say that a function $f(z_1, z_2, \ldots, z_n)$ of several complex variables is *holomorphic* if it is holomorphic in each variable separately: freeze all variables but one, and the function is holomorphic in the remaining variable. It can be shown (see [KRA1]) that such a function has a local power series expansion about each point of its domain. It also satisfies a suitable version of the Cauchy-Riemann equations.

If $P = (p_1, p_2, \ldots, p_n) \in \mathbb{C}^n$ and $r > 0$ then we define the *ball*

$$B(P, r) = \{(z_1, z_2) : |z_1 - p_1|^2 + |z_2 - p_2|^2 + \cdots + |z_n - p_n|^2 < r^2\}$$

and the *polydisc*

$$D^n(P, r) = \{(z_1, \ldots, z_n) : |z_j - p_j| < r \text{ for } j = 1, \ldots, n\} .$$

It is also useful to consider the *closed ball* and *closed polydisc*

$$\overline{B}(P, r) = \{(z_1, z_2) : |z_1 - p_1|^2 + |z_2 - p_2|^2 + \cdots + |z_n - p_n|^2 \leq r^2\} .$$

and

$$\overline{D}^n(P, r) = \{(z_1, \ldots, z_n) : |z_j - p_j| \leq r \text{ for } j = 1, \ldots, n\} .$$

Recall in one complex variable that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] .$$

This is just a new basis for the tangent space to the complex plane. We note particularly that

$$\begin{array}{ll} \frac{\partial}{\partial z} z = 1 & \qquad \frac{\partial}{\partial z} \overline{z} = 0 \\ \frac{\partial}{\partial \overline{z}} z = 0 & \qquad \frac{\partial}{\partial \overline{z}} \overline{z} = 1 . \end{array}$$

In several complex variables we write

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left[\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right] \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left[\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right]$$

for $j = 1, \ldots, n$. It is useful to define the differential operator on functions given by

$$\overline{\partial} u = \sum_j \frac{\partial u}{\partial \overline{z}_j} \, d\overline{z}_j . \tag{1.1}$$

The expression on the righthand side of equation (1.1) is called a *1-form.* If

$$\psi(z) = \sum_{j=1}^{n} \psi_j d\overline{z}_j$$

is any 1-form (of what we call *type* $(0,1)$), then we set

$$\overline{\partial}\psi = \sum_j \frac{\partial \psi_j}{\partial \overline{z}_k} \, d\overline{z}_k \wedge d\overline{z}_j \, .$$

It is easy to check that $\overline{\partial}(\overline{\partial}u) = \overline{\partial}^2 u = 0$ for any C^2 function u.

A domain Ω in $\mathbb{C}$ or $\mathbb{C}^n$ is said to be a *domain of holomorphy* if there is a holomorphic function g on Ω that cannot be analytically continued to any larger domain. [The full technical definition of "domain of holomorphy" is a bit more complicated, and we refer the reader to [KRA1] for the details.] It turns out that *every* domain in $\mathbb{C}$ is a domain of holomorphy. For fix such an $\Omega \subseteq \mathbb{C}$. Let $\mathcal{Z} = \{z_j\}$ be a collection of points in Ω that has no interior accumulation point but that accumulates at every boundary point of Ω. For instance, we may for each $k = 1, 2, \ldots$ let $S_k = \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) = 2^{-k}\}$. Now select a maximal collection $P_k \subseteq S_k$ of finitely many points that are about distance 2^{-k} apart. Set $\mathcal{Z} = \cup_k P_k$. Details of this construction may be found in [GRK]. Now, by Weierstrass's theorem, there is a holomorphic g on Ω that vanishes precisely at the points of $\mathcal{Z}$ and nowhere else. This g cannot be analytically continued to any larger open set, else its zeros would have an interior accumulation point and the function would be identically zero.

As we shall see below, one of the main points of the Hartogs theorem is that *not every domain in* $\mathbb{C}^2$ *is a domain of holomorphy.*

2 Statement of the Theorem and Consequences

The most classical rendition of the Hartogs extension phenomenon is as follows:

Theorem 1: Let

$$\Omega = D^2(0,2) \setminus \overline{D}^2(0,1) \, .$$

Figure 1 indicates what Ω looks like. Suppose that f is a holomorphic function on Ω. *We do not make any boundedness hypotheses about* f. Then there

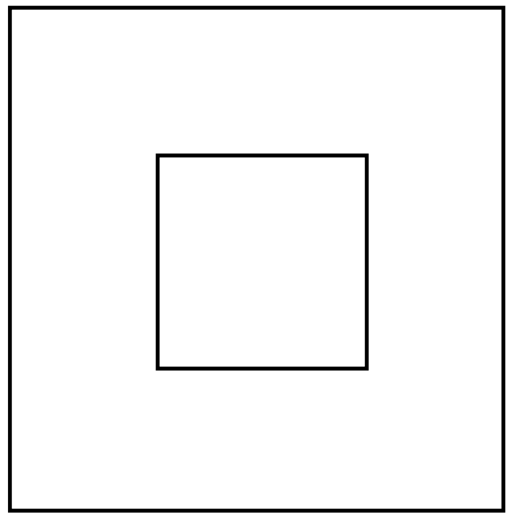

Figure 1: The Hartogs Extension Phenomenon.

is a holomorphic function F on $D^2(0,2)$ such that $F|_\Omega = f$.

One cannot help but think of the Riemann removable singularities theorem from one complex variable: If f is holomorphic on $D(0,r) \setminus \{0\}$ *and is bounded*, then f continues analytically to $D(0,1)$. The theorem above seems to be quite a lot stronger, since it hypothesizes f to be holomorphic on a domain with a big hole in the center (not just a point removed), it hypothesizes *nothing* about boundedness, and yet it derives the same conclusion.

At the risk of belaboring the obvious, one might try to construct a holomorphic function on Ω in the theorem that will not continue analytically. What about $f_1(z_1, z_2) = 1/z_1$ or $f_2(z_1, z_2) = 1/z_2$? The trouble with each of these functions is that its singularity is not just at the origin. We see that f_1 is singular on the entire complex line $\{z_1 = 0\}$ and f_2 is singular on the entire complex line $\{z_2 = 0\}$. In particular, neither function is holomorphic on Ω, so the theorem does not apply to them.

And in fact the theorem has a profound consequence that answers the question raised in the last paragraph:

Theorem 2: A holomorphic function of several complex variables cannot have an isolated singularity.

A companion result is

Theorem 3: A holomorphic function of several complex variables cannot have an isolated zero.

Let us now prove these theorems. For Theorem 2, suppose that f is holomorphic on a small deleted ball $B(P,r)\setminus\{P\}$, and that f has a singularity at P. We may find positive numbers $0 < r_1 < r_2 < r$ such that

$$D^2(P,r_2) \setminus \overline{D}^2(P,r_1) \subseteq B(P,r) \setminus \{P\} .$$

But then Theorem 1 applies (after a suitable translation and scaling of coordinates) on the domain $\Omega' \equiv D^2(P,r_2) \setminus \overline{D}^2(P,r_1)$, so that f continuous analytically across P. That is a contradiction.

Now let us look at Theorem 3. If f is holomorphic on some small ball centered at P and has an isolated zero at P, then $1/f$ has an isolated singularity at P. According to Theorem 2, that is impossible. And that ends the proof.

What we see—and this can be fleshed out by using some commutative algebra (see [KRA1, Ch. 6])—is that the zero set of a holomorphic function of n variables is in fact an $(n-1)$-dimensional variety, with some possibly singular subvarieties.

3 First Proof of the Hartogs Theorem

Let f be holomorpfic on Ω. For z_1 fixed, $|z_1| < 2$, we write

$$f_{z_1}(z_2) = f(z_1,z_2) = \sum_{j=-\infty}^{\infty} a_j(z_1) z_2^j, \tag{3.1}$$

where the coefficients of this Laurent expansion are given by

$$a_j(z_1) = \frac{1}{2\pi i} \oint_{|z_2|=3/2} \frac{f(z_1,\zeta)}{\zeta^{j+1}} d\zeta.$$

In particular, $a_j(z_1)$ depends holomorpfically on z_1 (by Morera's theorem, for instance). But $a_j(z_1) = 0$ for $j < 0$ and $1 < |z_1| < 2$. Therefore, by analytic continuation, $a_j(z_1) \equiv 0$ for $j < 0$. But then the series expansion (3.1) becomes

$$\sum_{j=0}^{\infty} a_j(z_1) z_2^j$$

and this series *defines* a holomorphic function F on all of $D^2(0,2)$ that agrees with the original function f on Ω. Thus Ω is *not* a domain of holomorphy—all holomorphic functions on Ω continue to the larger domain $D^2(0,2)$. This completes the proof of the "Hartogs extension phenomenon." □

4 More General Versions of the Hartogs Theorem

As previously indicated, one would like to replace the "polydisc with a smaller polydisc removed" by a more general domain. One possible formulation of the result is this:

> **Theorem:** Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let U be a neighborhood of the boundary of Ω. Suppose that f is holomorphic on $U \cap \Omega$. Then f analytically continues to all of Ω.

Unfortunately, as formulated here, this result is false. For if the boundary of Ω is disconnected, and if f is identically 1 on one piece of the boundary and identically 0 on the other piece of the boundary, then we would have a built-in contradiction.

So we certainly need to assume that the boundary of Ω is connected. But the boundary of Ω can still be geometrically very complicated. Any attempt at continuing f by sliding analytic discs—as in the original Hartogs proof—is going to run into complications (as Osgood discovered in [OSG]). It required a radical new methodology to address this issue, and we shall see what it is in the next section.

5 The Partial Differential Equations Approach

We begin with an interesting generalization of the Cauchy integral formula. Its proof is a straightforward application of Stokes's theorem, and we refer the reader to [KRA1] or [KRA2] for the details.

Proposition 4: If $\Omega \subseteq \mathbb{C}$ is a bounded domain with C^1 boundary and if $f \in C^1(\overline{\Omega})$ then, for any $z \in \Omega$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\Omega} \frac{\left(\partial f(\zeta)/\partial\overline{\zeta}\right)}{\zeta - z} d\overline{\zeta} \wedge d\zeta.$$

Now we have

Theorem 5: Let $\psi \in C^1_c(\mathbb{C})$. The function defined by

$$u(\zeta) = -\frac{1}{2\pi i}\int \frac{\psi(\xi)}{\xi - \zeta} d\overline{\xi} \wedge d\xi = -\frac{1}{\pi}\int \frac{\psi(\xi)}{\xi - \zeta}\, dV(\xi)$$

satisfies

$$\overline{\partial} u(\zeta) = \frac{\partial u}{\partial \overline{\zeta}}(\zeta) d\overline{\zeta} = \psi(\zeta) d\overline{\zeta}.$$

Proof: Let $D(0,R)$ be a large disc that contains the support of ψ. Then

$$\begin{aligned}
\frac{\partial u}{\partial \overline{\zeta}}(\zeta) &= -\frac{1}{2\pi i}\frac{\partial}{\partial \overline{\zeta}}\int_{\mathbb{C}} \frac{\psi(\xi)}{\xi - \zeta} d\overline{\xi} \wedge d\xi \\
&= -\frac{1}{2\pi i}\frac{\partial}{\partial \overline{\zeta}}\int_{\mathbb{C}} \frac{\psi(\xi + \zeta)}{\xi} d\overline{\xi} \wedge d\xi \\
&= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial \psi}{\partial \overline{\xi}}(\xi + \zeta)}{\xi} \delta\overline{\xi} \wedge d\xi \\
&= -\frac{1}{2\pi i}\int_{D(0,R)} \frac{\frac{\partial \psi}{\partial \overline{\xi}}(\xi)}{\xi - \zeta} \delta\overline{\xi} \wedge d\xi.
\end{aligned}$$

By Proposition 4, this last equals

$$\psi(\zeta) - \frac{1}{2\pi i}\int_{\partial D(0,R)} \frac{\psi(\xi)}{\xi - \zeta} d\xi = \psi(\zeta).$$

Here we have used the support condition on ψ. This is the result that we wished to prove. □

Now we have a version of this result in the several complex variables setting:

Theorem 6: Let $\psi = \sum_{j=1}^n \psi_j(z) d\overline{z}_j$ be a $(0,1)$ form on $\mathbb{C}^n$ with C^1_c coefficients and that is $\overline{\partial}-$closed. Then, for any choice of $j, 1 \leq j \leq n$, the function

$$u_j(z) = -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi_j(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)}{\zeta - z_j}\, d\overline{\zeta} \wedge d\zeta$$

satisfies $\overline{\partial} u_j = \psi$. For any j and $j', 1 \leq j, j' \leq n$, it holds that $u_j = u_{j'}$.

If $n > 1$ then the functions u_j are compactly supported; indeed, $u_j \equiv 0$ on the connected component of ${}^c(\operatorname{supp}\psi)$ that contains ∞.

Proof: To verify the claim about compact support, suppose for simplicity that $j = 1$. Set $u = u_1$. Now $\partial u/\partial \overline{z}_\ell = \psi_\ell = 0$, all ℓ, when z is large. Thus u is holomorphic in each variable separately, hence holomorphic, for z large—say when $|z| > R$. But, looking at the definition of u, we see that $u(z)$ itself must be zero when z is large. By analytic continuation, $u(z) = 0$ on the unbounded component of ${}^c(\cup_j \mathrm{supp}\psi_j)$ In other words, u is compactly supported. Given this, we note that the C^1 function $u_j - u_{j'}$ is annihilated by $\overline{\partial}$; thus it is holomorphic in each variable separately hence, because of its compact support, must be identically zero. It remains to prove the first assertion.

We write

$$\begin{aligned}
\frac{\partial}{\partial \overline{z}_\ell} u_j(z) &= -\frac{1}{2\pi i}\frac{\partial}{\partial \overline{z}_\ell}\int_{\mathbb{C}} \frac{\psi_j(z_1, \ldots, z_{j-1}, \zeta + z_j, z_{j+1}, \ldots, z_n)}{\zeta}\, d\overline{\zeta} \wedge d\zeta \\
&= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial \psi_j}{\partial \overline{z}_\ell}(z_1, \ldots, z_{j-1}, \zeta + z_j, z_{j+1}, \ldots, z_n)}{\zeta}\, d\overline{\zeta} \wedge d\zeta \\
&= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial \psi_\ell}{\partial \overline{z}_j}(z_1, \ldots, z_{j-1}, \zeta + z_j, z_{j+1}, \ldots, z_n)}{\zeta}\, d\overline{\zeta} \wedge d\zeta.
\end{aligned}$$

In this last equality we have exploited the compatibility condition $\overline{\partial}\psi = 0$. For fixed z, let $D(0, R)$ be a large disc in $\mathbb{C}$ that contains the support of $\psi(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n)$. Then the last integral can be written as

$$-\frac{1}{2\pi i}\int_{D(0,R)} \frac{\frac{\partial \psi_\ell}{\partial \overline{z}_j}(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)}{\zeta - z_j}\, d\overline{\zeta} \wedge d\zeta.$$

Now the argument that we used to complete the proof of Theorem 5 shows that this last expression equals $\psi_\ell(z)$. In other words,

$$\overline{\partial} u_j = \psi,$$

as was required. $\square$

Next is our most general version of the Hartogs extension phenomenon.

Theorem 7: Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain, $n > 1$. Let K be a compact subset of Ω with the property that $\Omega \setminus K$ is connected. If f is holomorphic on $\Omega \setminus K$ then there is a holomorphic F on Ω such that $F|_{\Omega\setminus K} = f$.

Proof: Let ϕ be a function in $C_c^\infty(\Omega)$ that is identically 1 on a neighborhood of K. Define

$$\widetilde{f}(z) = \begin{cases} (1-\phi(z)) \cdot f(z) & \text{if} \quad z \in \Omega \setminus K \\ 0 & \text{if} \quad z \in K. \end{cases}$$

Then $\widetilde{f} \in C^\infty(\Omega)$. Finally, set

$$\psi(z) = \overline{\partial}\widetilde{f}(z).$$

Then ψ satisfies the following crucial properties:

1. ψ has C^∞ coefficients;
2. $\overline{\partial}\psi = \overline{\partial}^2\widetilde{f} \equiv 0$;
3. $\operatorname{supp}\psi$ is a *compact* subset K_0 of Ω.

The first two of these properties are obvious and the last follows since $\widetilde{f}$ is holomorphic in $\Omega \cap$ (neighborhood of $\partial\Omega$).

By Theorem 6, there is a function $u \in C_c^\infty(\mathbb{C}^n)$, with support compact in Ω, so that $\overline{\partial}u = \psi$. In particular the function u is identically 0 in a neighborhood U of $\partial\Omega$. We define $F = \widetilde{f} - u$. Then

$$\overline{\partial}F = \overline{\partial}\widetilde{f} - \overline{\partial}u = \psi - \psi = 0$$

so F is holomorphic on Ω. Also, shrinking U if necessary,

$$F|_U = (\widetilde{f} - u)\Big|_U = \widetilde{f}\Big|_U = f|_U\,.$$

Therefore F agrees with f near $\partial\Omega$. Since $\Omega \setminus K$ is connected, we may conclude by the uniqueness of analytic continuation that $F = f$ on $\Omega \setminus K$. The proof is complete. □

The proof using partial differential equations, as presented here, is simple and natural. And it sidesteps all the topological difficulties which seemed to be present with a more elementary approach to the matter.

6 An Approach Using Spherical Harmonics

Now we use some ideas from the modern theory of harmonic analysis to give quite a different take on the Hartogs phenomenon.

Recall that a *spherical harmonic* is the restriction to the unit sphere in $\mathbb{R}^N$ of a harmonic polynomial. It can be shown (see [KRA3] or [STW]) that the spherical harmonics are very much like the trigonometric polynomials on the circle in $\mathbb{R}^2$. In particular, it can be shown that the spherical harmonics have closed linear span in L^2 equal to all L^2 functions.

For the purposes of using spherical harmonics in the contex of several complex variables, it is useful to consider *bigraded spherical harmonics*. This means that we classify a harmonic polynomial to be of *type* (p, q) if it is of degree p in the z variables and of degree q in the $\overline{z}$ variables. Thus we see in particular that a holomorphic polynomial is of type $(p, 0)$ and a conjugate holomorphic polynomial is of type $(0, q)$. Now we have

Theorem 8: Let $B = B(0, 1)$ denote the unit ball in $\mathbb{C}^n$. Let U be a neighborhood of ∂B. Suppose that f is a holomorphic function on $U \cap B$. Then there is a holomorphic F on B such that $F|_{U\cap B} = f|_{U\cap B}$.

Proof: Consider the bigraded spherical harmonic expansion of f on $S = \partial B$. Of course it will have only terms of type $(p, 0)$. Call them p_j. So

$$f\Big|_S = \sum_j a_j p_j \,,$$

where the a_j are complex coefficients. The series converges in the L^2 topology of ∂B. But then, by way of the standard Poisson integral formula, we see that the series also converges uniformly on compact subsets of B. And, since the partial sums are holomorphic, the full sum is a holomorphic function—in fact it is the function F that we seek. □

7 Even More General Versions of the Hartogs Phenomenon

It is natural to wonder, in Theorem 7, why we need to suppose that the function f is holomorphic in an entire neighborhood of $\partial\Omega$. Could we not begin with a function f defined *only on* $\partial\Omega$ and put some differential condition on f that would guarantee a holomorphic extension to the interior?

The answer is yes, and we briefly sketch some of the particulars (without proof) here. Let $\Omega \subseteq \mathbb{C}^n$ be a domain with C^1 boundary. As is customary in this subject, we write

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\} .$$

We mandate that $\nabla\rho \neq 0$ on $\partial\Omega$ and we call ρ a *defining function* for Ω (see [KRA1] for discussion of this concept).

Let f be a C^1 function on $\partial\Omega$. We say that f *satisfies the tangential Cauchy-Riemann equations* on $\partial\Omega$ if $\overline{\partial} f \wedge \overline{\partial}\rho = 0$ on $\partial\Omega$. Thus, in a sense, f satisfies the tangential Cauchy-Riemann equations if it is holomorphic in complex tangential directions on $\partial\Omega$. We often call such a function a CR function. Further details of these ideas are provided in [FOK] and [KRA4].

The fundamental result from this point of view is stated next. This theorem is generally attributed to Bochner [BOC], though it was anticipated by to work of Kneser [KNE]. A nice exposition of the matter appears in [HOR, p. 31].

Theorem 9: Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with C^1, connected boundary. Let f be a CR function on $\partial\Omega$. Then there is a function F, continuous on $\overline{\Omega}$, such that

- The function F agrees with f on $\partial\Omega$;
- The function F is holomorphic on Ω.

The proof of this result is a sophisticated exercise with the $\overline{\partial}$ operator—-certainly related to the proof of Theorem 7. We omit the details, but refer the reader to [HOR]. This is a very natural extension of the Hartogs phenomenon, for it treats the boundary more intrinsically.

It is interesting to note that a local version of Theorem 9 is in general false. That is to say, if U is a small open set that intersects $\partial\Omega$, and if f is a CR function on $U \cap \partial\Omega$, then it does not necessarily follow that f can be continued to a holomorphic function on one side of $U \cap \partial\Omega$.

8 A Metric Approach to Hartogs

In this section we present a point of view that is based in metric geometry, but that also has the flavor of real variables. The primary source for this idea is [KRA5], but the concept is developed a bit in [KRA6].

We begin by recalling the Kobayashi metric from geometric function theory. If Ω_1 and Ω_2 are domains, each in some complex space, then we define $\Omega_1(\Omega_2)$ to be the collection of all holomorphic maps from Ω_2 to Ω_1 (note that we follow the custom from cohomology theory for the order of the domains). As usual we let D denote the unit ball in $\mathbb{C}$. Now we have:

Definition 10 Let $\Omega \subseteq \mathbb{C}^n$ be open. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n$. The infinitesimal form of the *Kobayashi/Royden metric* is given by $F_K : \Omega \times \mathbb{C}^n \to \mathbb{R}$, where

$$\begin{aligned} F_K^\Omega(z,\xi) &\equiv \inf\{\alpha : \alpha > 0 \text{ and } \exists f \in \Omega(D) \text{ with } f(0) = z, (f'(0))\,(e_1) = \xi/\alpha\} \\ &= \inf\left\{\frac{|\xi|}{|(f'(0))(e_1)|} : f \in \Omega(D), (f'(0))(e_1) \text{ is a} \right. \\ &\qquad\qquad \left. \text{constant multiple of } \xi\right\} \end{aligned}$$

$$= \frac{|\xi|}{\sup\{|(f'(0))(e_1)| : f \in \Omega(B), (f'(0))(e_1) \text{ is a constant multiple of } \xi\}}.$$

One would like to think that the Kobayashi metric is complete on a smooth, pseudoconvex domain. And there is evidence that this is so (see, for instance, [FOLE]). By contrast, we have the following surprising result on a non-pseudoconvex domain (which domain is very much in the vein of the Hartogs extension phenomenon). See [KRA5].

Proposition 11 *Let*

$$\Omega = \left\{(z_1, z_2) : 1 < |z_1|^2 + |z_2|^2 < 4\right\}.$$

For $\epsilon > 0$*, let* $P_\epsilon = (-1-\epsilon, 0) \in \Omega$*. Then*

$$f_K^\Omega(P_\epsilon, (1,0)) \approx \epsilon^{-3/4}.$$

We cannot provide the details of the proof here; suffice it to say that it is a tricky calculation.

This proposition says in effect that the behavior of the normal derivative of a holomorphic function near a pseudoconcave boundary point is better than one might expect (the classical result of Hardy would anticipate ϵ^{-1} rather than $\epsilon^{-2/4}$—see [GOL]). As a result, using a classical argument as in [GOL] or [KRA7], one will find that the following is true.

Proposition 12 *Let* Ω *be as in the last proposition. Let* f *be any holomorphic function on* Ω*. Then* f *continues to be Lipschitz* $1/4$ *in a neighborhood of the boundary point* $(-1, 0)$ *in* $\partial\Omega$*.*

We see that this is, philosophically, in the vein of the Hartogs phenomenon. For it says that an arbitrary holomorphic function behaves much better in a neighborhood of a strongly pseudoconcave point than one might have expected. It does *not* anticipate the stronger result of analytic continuation, but it could be considered to be the real variable analogue.

9 Concluding Remarks

The Hartogs extension phenomenon is one of the most basic ideas in the function theory of several complex variables. It led naturally and immediately to the question of giving an extrinsic geometric characterization of domains of holomorphy. This became known as the *Levi problem*, and it occupied much of our attention in the first half of the twentieth century. It was ultimately solved by Oka, Bremerman, Narasimhan, Grauert, and others. The solution requires powerful machinery such as the Cousin problems or sheaf cohomology or partial differential equations. We certainly cannot provide the details here, but refer the reader to [KRA1] or [HOR].

It is always interesting and stimulating to have many approaches to an important and fundamental result. We have seen even in this short article how this multi-faceted approach can lead to new connections and new insights. The spherical harmonic approach presented here is new, and may lead to further investigations in the future. The metric/real variable approach in Section 8 is a definite novelty, and suggests a new approach to function theory that may even reveal new phenomena in one complex variable.

References

[BOC] S. Bochner, Analytic and meromorphic continuation by means of Green's formula, *Ann. Math.* 44(1943), 652–673.

[FOL] G. B. Folland, Spherical harmonic expansion of the Poisson-Szegö kernel for the ball, *Proc. Am. Math. Soc.* 47(1975), 401-408.

[FOK] G. B. Folland and J. J. Kohn, *The Neumann Problem for the Cauchy-Riemann Complex*, Princeton University Press, Princeton, NJ, 1972,

[FOLE] J. E. Fornæss and L. Lee, personal communication.

[GOL] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society, Providence, 1969.

[GRK] R. E. Greene and S. G. Krantz, *Function Theory of One Complex Variable*, 3rd ed., American Mathematical Society, Providence, RI, 2006.

[HAR] F. Hartogs, Zur Theorie der analytischen Functionen mehrener unabhangiger Veränderlichen insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten, *Math. Ann.* 62(1906), 1-88.

[HOR] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North Holland, Amsterdam, 1990.

[KNE] H. Kneser, *Monatshefte Math. und Phys.* 43(1936), 364–380.

[KRA1] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, RI, 2001.

[KRA2] S. G. Krantz, *Cornerstones of Geometric Function Theory: Explorations in Complex Analysis*, Birkhäuser Publishing, Boston, 2006.

[KRA3] S. G. Krantz, *A Panorama of Harmonic Analysis*, The Mathematical Association of America, Washington, D.C., 1999.

[KRA4] S. G. Krantz, *Partial Differential Equations and Complex Analysis*, CRC Press, Boca Raton, FL, 1992.

[KRA5] S. G. Krantz, On the boundary behavior of the Kobayashi metric, *Rocky Mountain J. Math.,* 22(1992).

[KRA6] S. G. Krantz, *Geometric Analysis and Function Spaces*, CBMS and the American Mathematical Society, Providence, 1993.

[KRA7] S. G. Krantz, Lipschitz spaces, smoothness of functions, and approximation theory, *Expositiones Math.* 3(1983), 193-260.

[MEP] J. Merker and E. Porten, A geometric proof of the Hartogs extension phenomenon, *Journal of Geometric Analysis*, to appear.

[OSG] W. F. Osgood, *Lehrbuch der Funktionentheorie*, vols. 1 and 2, B. G. Teubner, Leipzig, 1912.

[STW] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.