Higher Dimensional Conundra

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Abstract: In recent years, especially in the subject of harmonic analysis, there has been interest in geometric phenomena of $\mathbb{R}^N$ as $N \to +\infty$. In the present paper we examine several specific geometric phenomena in Euclidean space and calculate the asymptotics as the dimension gets large.

0 Introduction

Typically when we do geometry we concentrate on a specific venue in a particular space. Often the context is *Euclidean space*, and often the work is done in $\mathbb{R}^2$ or $\mathbb{R}^3$. But in modern work there are many aspects of analysis that are linked to concrete aspects of geometry. And there is often interest in rendering the ideas in Hilbert space or some other infinite dimensional setting. Thus we want to see how the finite-dimensional result in $\mathbb{R}^N$ changes as $N \to +\infty$.

In the present paper we study some particular aspects of the geometry of $\mathbb{R}^N$ and their asymptotic behavior as $N \to \infty$. We choose these particular examples because the results are surprising or especially interesting. We may hope that they will lead to further studies.

It is a pleasure to thank Richard W. Cottle for a careful reading of an early draft of this paper and for useful comments.

1 Volume in $\mathbb{R}^N$

Let us begin by calculating the volume of the unit ball in $\mathbb{R}^N$ and the surface area of its bounding unit sphere. We let Ω_N denote the former and ω_{N-1} denote the latter. In addition, we let $\Gamma(x)$ be the celebrated Gamma function of L. Euler. It is a helpful intuition (which is literally true when x is an

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integer) that $\Gamma(x) \approx (x-1)!$. We shall also use Stirling's formula which says that

$$k! \approx k^k \cdot e^{-k} \cdot \sqrt{2\pi k}$$

or, more generally,

$$\Gamma(x) \approx (x-1)^{x-1} e^{-(x-1)} \sqrt{2\pi(x-1)}$$

for $x \in \mathbb{R}$, $x > 0$.

Lemma 1 *We have that*

$$\int_{\mathbb{R}^N} e^{-\pi\|\mathbf{x}\|^2}\, d\mathbf{x} = 1.$$

Proof: The case $N = 1$ is familiar from calculus. We write

$$S = \int_{\mathbb{R}} e^{-\pi t^2}\, dt$$

hence

$$\begin{aligned} S^2 &= \int_{\mathbb{R}} e^{-\pi x^2}\, dx \int_{\mathbb{R}} e^{-\pi y^2}\, dy \\ &= \iint_{\mathbb{R}^2} e^{-\pi\|\mathbf{x}\|^2} d\mathbf{x} \\ &\overset{\text{(polar coordinates)}}{=} \int_0^\infty \int_{\|\mathbf{x}\|=1} e^{-\pi r^2} r\, ds(\mathbf{x}) dr \\ &= \omega_1 \frac{1}{2\pi} e^{-\pi r^2}\Big|_0^\infty \\ &= \frac{\omega_1}{2\pi} \\ &= 1 \end{aligned}$$

hence $S = 1$.

For the $N-$dimensional case, write

$$\int_{\mathbb{R}^N} e^{-\pi\|\mathbf{x}\|^2} dx = \int_{\mathbb{R}} e^{-\pi x_1^2} dx_1 \cdots \int_{\mathbb{R}} e^{-\pi x_N^2} dx_N$$

and apply the one-dimensional result. □

Let σ be the unique rotationally invariant area measure on $S_{N-1} = \partial B_N$.

Lemma 2 *We have*

$$\omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}\,.$$

Proof: Introducing polar coordinates we have

$$1 = \int_{\mathbb{R}^N} e^{-\pi|x|^2} dx = \int_{S^{N-1}} d\sigma \int_0^\infty e^{-\pi r^2} r^{N-1} dr$$

or

$$\frac{1}{\omega_{N-1}} = \int_0^\infty e^{-\pi r^2} r^N \frac{dr}{r}.$$

Letting $s = r^2$ in this last integral and doing some obvious manipulations yields the result. $\square$

Corollary 3 *The volume of the unit ball in* $\mathbb{R}^N$ *is*

$$\Omega_N = \frac{\omega_{N-1}}{N} = \frac{2\pi^{N/2}}{\Gamma(N/2)\cdot N}\,.$$

Proof: We calculate that

$$\Omega_N = \iint\limits_B 1\,dV(x) \stackrel{\text{(polar coordinates)}}{=} \int_0^1 \int_{\|\mathbf{x}\|=1} 1\cdot r^{N-1}\, d\sigma(\mathbf{x}) dr = \omega_{N-1}\cdot \left.\frac{r^N}{N}\right|_0^1 = \frac{\omega_{N-1}}{N}\,.$$

That completes the proof. $\square$

All other things being equal, we might suppose that the volume $\Omega(N)$ of the unit ball in N-dimensional Euclidean space increases with N. This is in fact true for $1 \le N \le 5$, as we may verify by direct calculation. After that the volume of the unit ball decreases monotonically (see [COT]). In fact we have the remarkable result that the volume of the Euclidean unit ball in N-space tends to 0 as $N \to \infty$. More formally,

Proposition 4 *We have the limit*

$$\lim_{N\to+\infty} \Omega(N) = 0\,.$$

Proof: We calculate that

$$\begin{aligned}
\text{(Volume of Unit Ball)} &= \frac{2\pi^{N/2}}{\Gamma(N/2)\cdot N} \\
&\approx \frac{2\pi^{N/2}}{((N-2)/2)^{(N-2)/2}e^{-(N-2)/2}\sqrt{2\pi[(N-2)/2]}\cdot N} \\
&\approx \frac{(2\pi e)^{N/2}\cdot 2}{N^{(N-1)/2}\cdot\sqrt{\pi}\cdot N} \\
&\approx \frac{(2\pi e)^{N/2}\cdot 2}{N^{(N+1)/2}\cdot\sqrt{\pi}} \\
&\approx \left(\frac{2\pi e}{N}\right)^{N/2}\cdot\frac{1}{\sqrt{N}}\cdot\frac{2}{\sqrt{\pi}}\,.
\end{aligned}$$

This expression clearly tends to 0 as $N \to +\infty$. □

In fact we can actually say something about the *rate* at which the volume of the ball tends to zero. We have

Proposition 5 *We have the estimate*

$$0 \le \Omega_N \le 2\cdot\frac{20^{N/2}}{N^{(N+1)/2}}\,.$$

Proof: Follows by inspection of the last line of the proof of Proposition 4. □

In fact something more is true about the volumes of balls in high-dimensional Euclidean space.

Proposition 6 *Let $R > 0$ be fixed. Then*

$$\lim_{N\to+\infty} Vol(B(0,R)) = 0\,.$$

In other words, the volume of the ball of radius R tends to 0.

Proof: From the formula for the volume of the unit ball we have that

$$\lim_{N\to+\infty}\mathrm{Vol}(B(0,R)) = \lim_{N\to+\infty}\left(\frac{2\pi e R^2}{N}\right)^{N/2}\cdot\frac{1}{\sqrt{N}}\cdot\frac{2}{\sqrt{\pi}}\,.$$

This expression clearly tends to 0 as $N \to +\infty$. $\square$

We leave the proof of the next result as an exercise for the reader; simply examine the formula for ω_{N-1}:

Proposition 7 *Let $R > 0$. Then the surface area of the sphere of radius R in $\mathbb{R}^N$ tends to 0 as $N \to +\infty$.*

For comparison purposes, we may make these very simple observations:

- **(a)** The volume of the unit cube in $\mathbb{R}^N$ is 1 for any N.
- **(b)** The surface area of the boundary of the unit cube in $\mathbb{R}^N$ is $2N$, which tends to $+\infty$ as $N \to +\infty$.
- **(c)** The volume of the cube of side 2 in $\mathbb{R}^N$ (note that the unit ball has diameter 2) is 2^N. This quantity tends to $+\infty$ as $N \to +\infty$.
- **(d)** The surface area of the boundary of the cube of side 2 in $\mathbb{R}^N$ is $2N \cdot 2^{N-1}$. This quantity of course tends to $+\infty$ as $N \to +\infty$.

We conclude that the geometry of the ball is very different from the geometry of the cube, and the difference becomes more pronounced as the dimension of the ambient space increases.

The following very simple but remarkable fact comes up in considerations of spherical summation of Fourier series.

Proposition 8 *As $N \to +\infty$, the volume of the unit ball in $\mathbb{R}^N$ is concentrated more and more out near the boundary sphere. More precisely, let $\delta > 0$. Then*

$$\lim_{N \to +\infty} \frac{\text{volume}(B(0,1) \setminus B(0, 1-\delta))}{\text{volume}(B(0,1)} = 1\,.$$

Proof: We have

$$\begin{aligned}
\lim_{N \to +\infty} \frac{\text{volume}(B(0,1) \setminus B(0, 1-\delta))}{\text{volume}(B(0,1)} &= \lim_{N \to +\infty} \frac{[1 - (1-\delta)^N] \cdot [2\pi^{N/2}]/[\Gamma(N/2) \cdot N]}{[2\pi^{N/2}]/[\Gamma(N/2) \cdot N]} \\
&= \lim_{N \to +\infty} 1 - (1-\delta)^N \\
&= 1\,.
\end{aligned}$$

That is the desired conclusion. $\square$

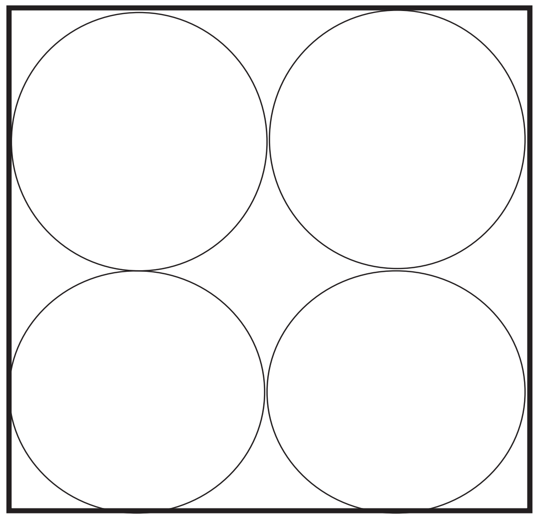

Figure 1: The configuration in dimension 2.

2 A Case of Leakage

The title of this section gives away the punchline of the example. Or so it may seem to some.

Consider at first a square box of side 4 with center at the origin 0 and sides parallel to the coordinate axes in the Euclidean plane. We may inscribe in this box four discs of radius 1, as shown in Figure 1. These discs will be called *primary discs*. Once those four discs are inscribed, we may inscribe a small, shaded disc in the middle as shown in Figure 2. We set

$$\mathcal{R}_2 = \frac{\text{area of shaded disc}}{\text{area of large box}} .$$

The same construction may be performed in Euclidean dimension 3. Examine Figure 3. It suggests a rectangular parallelepiped with center at the origin, all sides parallel to the coordinate planes and equal to 4, and 8 unit balls inscribed inside in a canonical fashion. These eight primary balls determine a unique inscribed shaded ball in the center. We set

$$\mathcal{R}_3 = \frac{\text{volume of shaded ball}}{\text{volume of large box}} .$$

A similar construction may be performed in any dimension $N \geq 2$, with

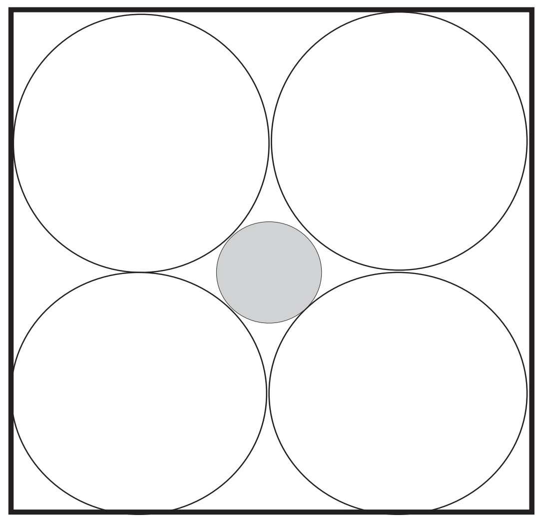

Figure 2: The shaded disc in dimension 2.

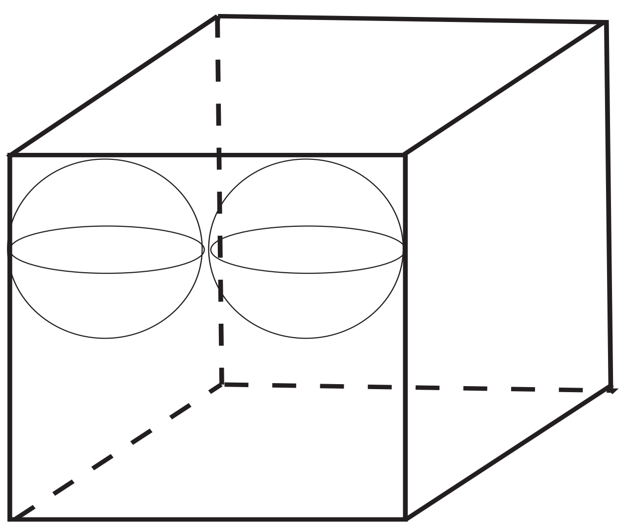

Figure 3: The configuration in dimension 3.

2^N unit balls inscribed in a rectangular box of side 4. The ratio $\mathcal{R}_N$ is then calculated in just the same way. The question is then

What is the limit $\lim_{N\to\infty} \mathcal{R}_N$ as $N \to +\infty$?

It is natural to suppose, and most people do suppose, that this limit (assuming it exists) is between 0 and 1. All other things being equal, it is likely equal to either 0 or 1. Thus it comes as something of a surprise that this limit is in fact equal to $+\infty$. Let us now enunciate this result and prove it.

Proposition 9 *The limit*

$$\lim_{N\to+\infty} \mathcal{R}_N = +\infty\,.$$

Of course this result is counter-intuitive, because we all instinctively believe that the shaded ball, in any dimension, is contained inside the big box. Such is not the case. We are being fooled by the 2-dimensional situation depicted in Figure 1. In that special situation, any of the two adjacent primary discs actually touch in such a way as to trap the shaded disc in a particular convex subregion of the big box (see Figure 4). So certainly it must be that $\mathcal{R}_2 < 1$. But such is not the case in higher dimensions. There is actually a gap on each side of the box through which the shaded ball can leak. And indeed it does.

This is what we shall now show. First we shall perform the calculation of $\mathcal{R}_N$ for each N and confirm that the expression tends to $+\infty$ as $N \to +\infty$. Then we shall calculate the first dimension in which the shaded ball actually leaks out of the box.

Proof of the Proposition: Notice that the center of one of the primary balls is at the point $(1, 1, \ldots, 1)$. It is a simple matter to calculate that a boundary point of this ball that is nearest to the center of the box is located at $P^* \equiv (1 - 1/\sqrt{N}, 1 - 1/\sqrt{N}, \ldots, 1 - 1/\sqrt{N})$. Since the shaded ball will osculate the primary ball at that point, we see that the shaded ball has center the origin and radius equal to

$$\text{dist}(0, P^*) = \sqrt{(1 - 1/\sqrt{N})^2 + (1 - 1/\sqrt{N})^2 + \ldots + (1 - 1/\sqrt{N})^2} = \sqrt{N + 1 - 2\sqrt{N}}\,.$$

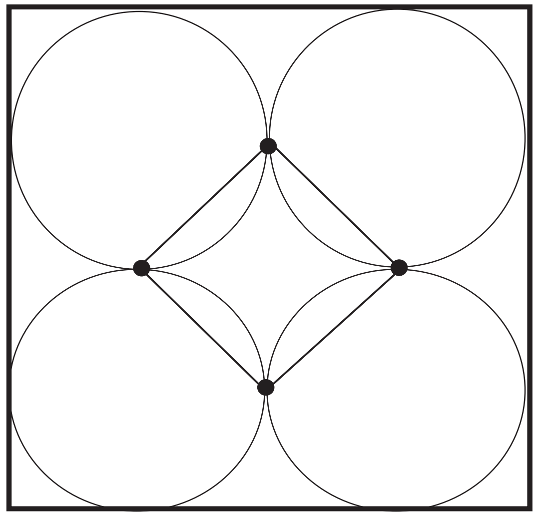

Figure 4: The disc trapped in dimension 2.

Thus we see that the volume of the shaded ball is

$$[N + 1 - 2\sqrt{N}]^{N/2} \cdot \Omega_N \,.$$

The ratio $\mathcal{R}_N$ is then

$$\mathcal{R}_N = \frac{[N + 1 - 2\sqrt{N}]^{N/2} \cdot \Omega_N}{4^N} \,.$$

Now we may simplify this last expression to

$$\mathcal{R}_N = \frac{2 \cdot \pi^{N/2}}{\Gamma(N/2) \cdot N} \cdot \frac{[N + 1 - 2\sqrt{N}]^{N/2}}{4^N} \,.$$

After some simplification we find that

$$\mathcal{R}_N = \frac{2(\pi/16)^{N/2} \cdot [N + 1 - 2\sqrt{N}]^{N/2}}{\Gamma(N/2) \cdot N} \,.$$

By Stirling's formula, this last expression is approximately equal to

$$\frac{2 \cdot (\pi/16)^{N/2}[N + 1 - 2\sqrt{N}]^{N/2}}{N} \cdot \left(\frac{N-2}{2}\right)^{(2-N)/2} \cdot e^{(N-2)/2} \cdot \frac{1}{\sqrt{\pi(N-2)}}$$

$$= \frac{2}{N}\left[\frac{\pi}{16}(N + 1 - 2\sqrt{N}) \cdot \frac{2}{N-2} \cdot e\right]^{N/2} \cdot \frac{N-2}{2} \cdot \frac{1}{\sqrt{\pi(N-2)}} \cdot \frac{1}{e} \,.$$

After some manipulation, we finally find that

$$\begin{aligned}\lim_{N\to\infty}\mathcal{R}_N &= \lim_{N\to\infty}\frac{2}{N}\left(\frac{\pi e}{8}\right)^{N/2}\cdot\left(\frac{N+1}{N-2}\right)^{N/2}\cdot\frac{N-2}{2}\cdot\frac{1}{\sqrt{\pi(N-2)}}\cdot\frac{1}{e}\\ &= \lim_{N\to\infty}\frac{2}{N}\left(\frac{\pi e}{8}\right)^{N/2}\left(1+\frac{3}{N-2}\right)^{N/2}\cdot\frac{N-2}{2}\cdot\frac{1}{\sqrt{\pi(N-2)}}\cdot\frac{1}{e}\,.\end{aligned}$$

Now, in the limit, we may replace expressions like $N-2$ by N. And we may reparametrize N as $3N$. The result is

$$\begin{aligned}&\lim_{N\to\infty}\left(\frac{\pi e}{8}\right)^{3N/2}\cdot\left(1+\frac{1}{N}\right)^{3N/2}\cdot\frac{1}{\sqrt{\pi(3N-2)}}\cdot\frac{1}{e}\\ &= \lim_{N\to\infty}\left(\frac{\pi e}{8}\right)^{3N/2}\left[\left(1+\frac{1}{N}\right)^{N}\right]^{3/2}\cdot\frac{1}{\sqrt{\pi(3N-2)}}\cdot\frac{1}{e}\,.\end{aligned}$$

What we see now is that this last equals

$$\lim_{N\to\infty}\left(\frac{\pi e}{8}\right)^{3N/2}\cdot\frac{\sqrt{e}}{\sqrt{3N\pi}}\,.$$

Plainly, because $\pi e/8 > 1$, this limit is $+\infty$. That proves the result. □

And now we turn to the question of when the shaded ball starts to leak out of the big box. This is in fact easy to analyze. We need only determine when the radius of the shaded ball exceeds 1. First notice that the radius of the shaded ball is monotone increasing in N. Now we need to solve

$$\sqrt{N+1-2\sqrt{N}} > 1\,.$$

This is a simple algebra problem, and the solution is $N > 4$. Thus, beginning in dimension 5, the shaded ball will "leak out of" the large box. It may be noted that Richard W. Cottle has made a study of mathematical phenomena that change (in the manner of a catastrophe—see [ZEE]) between dimensions 4 and dimensions 5. The results may be found in [COT].

A final remark concerns the components of our analysis. For fixed dimension N, the volume of the box is $V_B = 4^N$. But the volume of the union of the primary balls is

$$V_P = 2^N \cdot \Omega_N = \frac{2^N \cdot 2\pi^{N/2}}{\Gamma(N/2) \cdot N} \approx \frac{2^N \cdot 2\pi^{N/2}}{N} \cdot \left(\frac{N-2}{2}\right)^{(2-N)/2} \cdot e^{(N-2)/2} \cdot \frac{1}{\sqrt{\pi(N-2)}} \, .$$

This expression of course tends to 0. The result makes sense, since the volume of the shaded ball is exploding.

3 Centroids

This final section of the paper will be more like an invitation to further exploration. We cannot include all the details of the calculations, as they are too recondite and complex. Yet the topic is very much in the spirit of the theme of this paper, and we cannot resist including a few pointers to this new and interesting work (for which see [KRA1] and [KRMP]).

The inspiration for this work is the following somewhat surprising observation. Let T be a triangle in the plane (see Figure 5). There are three ways to calculate the centroid of this figure: **(i)** average the vertices, **(ii)** average the edges, or **(iii)** average the 2-dimensional solid figure. And the question is: are these three versions of the centroid the same? The answer is that **(i)** and **(iii)** are *always the same*. Generically **(ii)** is different. In fact the three versions of the centroid coincide if and only if the triangle is equilateral [KRMP].

We used this fact as a springing-off point to investigate analogous questions in higher dimensions. Consider the simplex $\mathbf{S}$ in $\mathbb{R}^N$ that is the convex hull of the points $0 = (0, 0, \ldots, 0)$, $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 0, 1)$. Refer to Figure 6. Such an N-dimensional geometric figure comes equipped with $(N+1)$ notions of centroid: we can average the vertices (or 1-dimensional skeleton) $\mathcal{S}_0$, or we can average over the 1-dimensional skeleton $\mathcal{S}_1$, or we can average over the two-dimensional skeleton $\mathcal{S}_2$A, or ... we can average over the $(N-1)$-dimensional skeleton $\mathcal{S}_{N-1}$, or we can average over the N-dimensional solid $\mathcal{S}_N$. There results the centroids $\mathcal{C}_{0,N}$, $\mathcal{C}_{1,N}$, $\ldots$, $\mathcal{C}_{N,N}$. And the question is: Are these different notions of centroid all the same? And here is the somewhat surprising answer:

> In dimensions 2 through 12 (for the ambient space), the skeletons $\mathcal{S}_0$ and $\mathcal{S}_N$ for the simplex $\mathbf{S}$ have the same centroid. In those same dimensions, the skeletons $\mathcal{S}_1$, $\mathcal{S}_2$, $\ldots$, $\mathcal{S}_{N-1}$ all have different

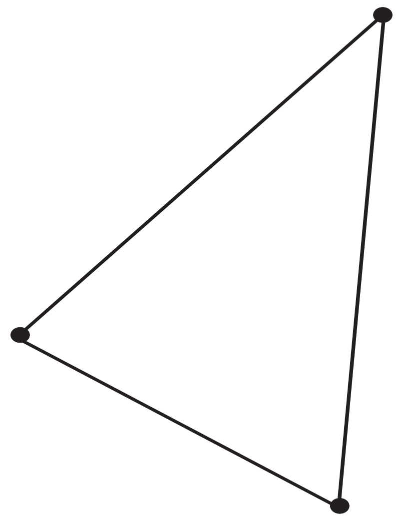

Figure 5: Centroids for a triangle.

> centroids, and the centroids all differ from the common centroid for $\mathcal{S}_0$ and $\mathcal{S}_N$. But in dimension 13 things are different. In fact in that dimension the skeletons $\mathcal{S}_3$ and $\mathcal{S}_8$ have the same centroid.

Let us say a word about why these facts are true. Let $\mathbf{e}_j$ denote the j^{th} coordinate vector in $\mathbb{R}^N$ (i.e., the vector with a 1 in the j^{th} position and 0s in all other slots). Then a sophisticated computation with elementary calculus yields that the centroid of the k-skeleton $\mathcal{S}_k$ of the simplex which is the convex hull of $0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ is

$$\mathcal{C}_{k,N} = \frac{1}{N} \cdot \frac{k + (N-k)\sqrt{k+1}}{(k+1) + (N-k)\sqrt{k+1}} (\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_N) .$$

From this formula it can immediately be verified that

$$\mathcal{S}_0 = \mathcal{S}_N = \frac{1}{N+1} (\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_N) .$$

It can also be checked that, in dimensions 2 through 12, all the intermediate skeletons have distinct centroids. But, in dimension $N = 13$, we observe that

$$\mathcal{C}_{3,13} = \mathcal{C}_{8,13} = \frac{23}{13 \cdot 24} (\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_N) .$$

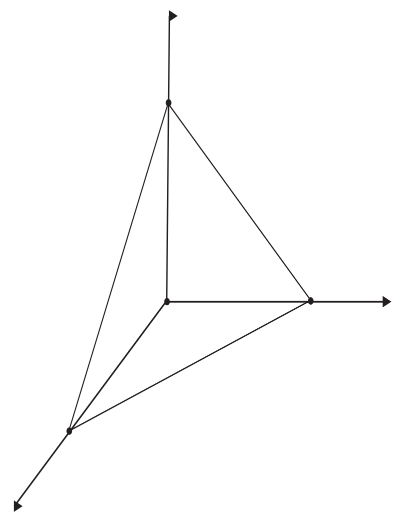

Figure 6: A simplex in $\mathbb{R}^N$.

We may well ask whether dimension $N = 13$ is the only dimension in which there are two intermediate skeletons with the same centroid. The answer is "no"; there are in fact infinitely many such dimensions (although they are quite sparse—sparser than the prime integers). We may verify this assertion by using the following Diophantine formula.

Theorem 10 *Fix a dimension $N \geq 2$. Consider the simplex $\mathbf{S}$ as described above. There are skeletons of dimension k_1 and k_2, $1 \leq k_1 < k_2 \leq N-1$, of the simplex $\mathbf{S}$ which have the same centroid if and only if $k_1 = a^2 - 1$, $k_2 = b^2 - 1$ (for positive integers a and b) and, in addition,*

$$N = (b^2 + ab + a^2) - (b + a) - 1\,. \tag{$\star$}$$

Obviously this theorem gives us a tool for finding dimensions in which the simplex $\mathbf{S}$ has two intermediate skeletons with the same centroid. The following table gives some values of the dimension, and of the intermediate dimensions of skeletons which have the same centroid. Of course this data may be confirmed by direct calculation with the formula ($\star$). We stress that there are in fact infinitely many dimensions in which this phenomenon occurs. The proof of this statement is a nontrivial exercise in elementary number theory (see [KRMP]).

value of N	**value of k_1**	**value of k_2**	**approx. coord. of centroid**
13	3	8	0.0737179487
21	3	15	0.0464285714
29	8	15	0.0340038314
31	3	24	0.0317204301
40	8	24	0.0247619047
43	3	35	0.0229789590
51	15	24	0.0194852941
53	8	35	0.0187368973
57	3	48	0.0173872180
65	15	35	0.0153133903

We conclude this discussion by recording the fact that it is *impossible* in any dimension for there to be three intermediate skeletons with the same centroid.

Proposition 11 *For no dimension N can there exists 3 distinct number $1 \le k_1 < k_2 < k_3 \le N-1$ such that the centroids $\mathcal{C}_{k_1,N}$, $\mathcal{C}_{k_2,N}$, $\mathcal{C}_{k_3,N}$ for the simplex* **S** *coincide.*

Proof: We let

$$Q(a,b) = (b^2 + ab + a^2) - (b + a) - 1 \, .$$

It suffices for us to show that there do not exist natural numbers $a < b < c$ such that $Q(a,b) = Q(a,c)$. Seeking a contradiction, we suppose that such a triple does indeed exist.

Then

$$b^2 + ab - b = c^2 + ac - c$$

or

$$b^2 + (a-1)b = c^2 + (a-1)c \, .$$

Since $a \ge 1$, the function $b \mapsto b^2 + (a-1)b$ is strictly increasing, which yields a contradiction. □

The exploration of centroids for simplices of high dimension is a new venue of inquiry. There are many new phenomena, and more to be discovered. See [KRMP] for more results along these lines. The reference [ZON] is also of interest.

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