Pseudoconvexity, Analytic Discs, and Invariant Metrics

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Abstract: *We begin by studying characterizations of pseudoconvexity, and also of finite type, using analytic discs. The results presented are analogous to well-known ideas from the real variable setting in which "pseudoconvex" is replaced by "convex" and "analytic disc" is replaced by "line segment".*

The second part of the paper concerns regularity results for the Kobayashi metric. Of course this metric is defined using analytic discs, so the discussion is a natural extension of that in the first part of the paper. We also comment on the Carathéodory metric.

0 Introduction

Convexity is a classical idea. Archimedes used a version of convexity in his considerations of arc length. Yet the idea was not formalized until 1934 in the monograph of Bonneson and Fenchel [BOF].

The classical definition of convexity is this: An open domain $\Omega \subseteq \mathbb{R}^N$ is convex if, whenever $P, Q \in \Omega$, then the segment $\overline{PQ}$ connecting P to Q lies in Ω. We call this the *synthetic* definition of convexity. It has the advantage of being elementary and accessible (see [VAL]). The disadvantages are that it is non-quantitative and non-analytic. It is of little use in situations of mathematical analysis where it is most likely to arise.

The *analytic* definition of convexity is a bit more recondite. Let $\Omega \subseteq \mathbb{R}^N$ have C^2 boundary. For us this means that there exists a C^2 function ρ defined

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in a neighborhood U of $\partial\Omega$ such that

$$\Omega \cap U = \{x \in U : \rho(x) < 0\}$$

and further that $\nabla\rho \neq 0$ on $\partial\Omega$. We call ρ a *defining function* for Ω. Let $P \in \partial\Omega$. We say that a vector $w \in \mathbb{R}^N$ is a *tangent vector* to $\partial\Omega$ at P, and we write $w \in T_P(\partial\Omega)$, if

$$\sum_{j=1}^{N} \frac{\partial \rho}{\partial x_j}(P) w_j = 0 \, .$$

The domain Ω is said to be *analytically convex* at P if

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k \geq 0 \tag{1}$$

for all $w \in T_P(\partial\Omega)$.

A moment's thought reveals that the condition (1) simply mandates that the second partial derivative of ρ in the direction w be nonnegative. This is the classical "convex up" condition from calculus. This analytic definition of convexity has the advantage that it can be localized to individual boundary points, and it is *quantitative*. It is a straightforward exercise (see [KRA1, pp. 122–123]) to see that the analytic definition of convexity is equivalent to the synthetic definition of convexity, at least for domains with C^2 boundary.

The notion of pseudoconvexity has a slightly different ontology. Discovered by E. E. Levi in the study of domains of holomorphy, this idea was first formulated in its analytic form. Let $\Omega \subseteq \mathbb{C}^n$ have C^2 boundary. Let ρ be a C^2 defining function for Ω as in our earlier discussion of convexity. Let $P \in \partial\Omega$. We say that $\xi \in \mathbb{C}^n$ is a *complex tangent vector* at P, and we write $\xi \in \mathcal{T}_P(\partial\Omega)$, if

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P) \xi_j = 0 \, .$$

The point P is said to be a point of *Levi pseudoconvexity* if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(P) \xi_j \overline{\xi}_k \geq 0 \tag{2}$$

for all $\xi \in \mathcal{T}_P(\partial\Omega)$.

It is not a simple matter to give an elementary geometrical interpretation to the expression (2). Part of the purpose of the present paper is to come to some basic geometric understanding of this notion of pseudoconvexity.

It is appropriate to record in passing a classical, alternative notion of pseudoconvexity. Let $\Omega \subseteq \mathbb{C}^n$ be any domain (smoothly bounded or not). We say that Ω is *Hartogs pseudoconvex* if, with δ_Ω denoting the function of Euclidean distance to the boundary, we have that $-\log \delta_\Omega$ is plurisubharmonic on Ω. It is known—see [KRA1, p. 144]—that a domain Ω with C^2 boundary is Levi pseudoconvex if and only if it is Hartogs pseudoconvex.

While the notion of Hartogs provides a sort of synthetic idea of pseudoconvexity, it is not strictly analogous to the idea that is used in classical convexity theory. Convexity has played an increasingly prominent role in the function theory of several complex variables in recent years (see [LEM] and [MCN1], [MCN2]). Thus it is worthwhile to be able to develop in further detail the analogy between classical convexity theory and modern pseudoconvexity theory. That is our purpose in the present paper.

In the development of Sections 1–3, we shall certainly see several points of contact with classical ideas (see [KRA1], especially Theorem 3.3.5 therein). The significance of this point of view for pseudoconvexity has been borne out in many contexts, particularly in the study of automorphism groups. See especially [GRK2] where this approach is used decisively.

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1 Analytic Discs and Pseudoconvexity

The results that we present here have a history. Certainly they are related to the classical *Kontinuitätssatz*, for which see [KRA1, p. 144]. But the proofs, of necessity, are different.

Let $D \subseteq \mathbb{C}$ be the unit disc. An *analytic disc* in $\mathbb{C}^n$ is a nonconstant holomorphic mapping $\varphi : D \to \mathbb{C}^n$. A *closed analytic disc* in $\mathbb{C}^n$ is a continuous mapping $\psi : \overline{D} \to \mathbb{C}^n$ such that $\psi\big|_D$ is holomorphic. In practice we may refer to either of these simply as an "analytic disc". The boundary of a closed analytic disc is just $\psi(\partial D)$. It will frequently be convenient to confuse the mapping φ or ψ with the image disc $\varphi(D)$ or $\psi(\overline{D})$. We do so without further comment. The center of an analytic disc is $\varphi(0)$ or $\psi(0)$.

In this paper we shall think of the boundary of a closed analytic disc as the complex-analytic analogue of two points P and Q in the classical theory of convex sets. We shall think of the (image) analytic disc $\psi(\overline{D})$ as the complex-analytic analogue of the segment $\overline{PQ}$ that connects P and Q in the real-variable context.

Thus we should like to have a characterization of pseudoconvexity, in terms of analytic discs, that is parallel to the synthetic characterization of convexity in terms of segments. It is the following.

Proposition 1 *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with C^2 boundary. Then Ω is pseudoconvex in the classical sense if there is a number $\delta_0 > 0$ so that, whenever $\psi : \overline{D} \to \mathbb{C}^n$ is a closed analytic disc in $\mathbb{C}^n$ with diameter less than δ_0, and if $\psi(\partial D) \subseteq \partial\Omega$, then $\psi(\overline{D}) \subseteq \overline{\Omega}$.*

Proof: Let "dist" denote Euclidean distance. Choose $\epsilon_0 > 0$ so that

$$U_{\epsilon_0} \equiv \{z \in \mathbb{C}^n : \mathrm{dist}(z, \partial\Omega) < \epsilon_0\}$$

is a tubular neighborhood of $\partial\Omega$ (see [HIR]). Let $\delta_0 = \epsilon_0/100$. Let ψ be a closed analytic disc as in the statement of the proposition. It follows immediately that the (image of the) closed analytic disc lies entirely inside the tubular neighborhood U_{ϵ_0}.[2] Now there are two cases:

(a) **Some point of $\psi(D)$ lies outside $\overline{\Omega}$.** In this case let $p_0 \equiv \psi(\zeta_0)$ be the point of $\psi(D)$ that lies furthest from $\partial\Omega$. Let ν be the unique normal vector from $\partial\Omega$ out to p_0. Say that ν emanates from the base point $q_0 \in \partial\Omega$. Then the domain

$$\widehat{\Omega} \equiv \Omega + \nu = \{z + \nu : z \in \Omega\}$$

has the property that the disc $\psi(D)$ is tangent to $\partial\widehat{\Omega}$ at p_0 and the punctured disc $\psi(D) \setminus \{p_0\}$ lies entirely in $\widehat{\Omega}$. But of course $\widehat{\Omega}$ is pseudoconvex with C^2 boundary. So this last is impossible (see [KRA1, p. 144]). We have eliminated this case.

(b) **All points of $\psi(D)$ lie in Ω.** In this case $\psi(\overline{D}) \subseteq \overline{\Omega}$ and we are done.

[2]The purpose of forcing the analytic disc to lie inside a tubular neighborhood is to guarantee that the disc does not form the basis of a homology class in the boundary.

□

An argument similar to the one just presented, but even simpler, gives the following result. It is closer to the spirit of the classical synthetic definition of convexity.

Proposition 2 *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with C^2 boundary. Then Ω is pseudoconvex in the classical sense if there is a number $\delta_0 > 0$ so that, whenever $\psi : \overline{D} \to \mathbb{C}^n$ is a closed analytic disc in $\mathbb{C}^n$ with diameter less than δ_0, and if $\psi(\partial D) \subseteq \Omega$, then $\psi(\overline{D}) \subseteq \Omega$.*

Yet another variant is this:

Proposition 3 *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with C^2 boundary. Then Ω is pseudoconvex in the classical sense if there is a number $\delta_0 > 0$ so that, whenever $\psi : \overline{D} \to \mathbb{C}^n$ is a closed analytic disc in $\mathbb{C}^n$ with diameter less than δ_0, and if $\psi(\partial D) \subseteq \overline{\Omega}$, then $\psi(\overline{D}) \subseteq \overline{\Omega}$.*

2 The Concept of Finite Type

The idea of finite type was first conceived in the paper [KOH] of Kohn. Kohn's idea was to measure the complex-analytic flatness of a boundary point of a domain in $\mathbb{C}^2$; this was conceived as an obstruction to subelliptic estimates for the $\overline{\partial}$-Neumann problem.

Later, Bloom and Graham [BLG] generalized Kohn's work to higher dimensions. Perhaps more significantly, they isolated two very interesting definitions of finite type and proved them to be equivalent. We now briefly review these two definitions.

Definition 1 Let $\Omega \subseteq \mathbb{C}^n$ be a domain with C^∞ boundary. Let ρ be a smooth defining function for Ω. Let $P \in \partial\Omega$. Let m be a positive integer. We say that P has *geometric type* at least m if there is a nonsingular (for us nonsingular means that $\varphi'(0) \neq 0$) analytic disc $\varphi : D \to \mathbb{C}^n$ such that $\varphi(0) = P$ and

$$|\rho(\varphi(\zeta))| \leq C|\zeta|^m . \tag{3}$$

The greatest m for which this is true is called the *type* of the point P. If there is no greatest m then the point P is said to be of *infinite type*.

Of course a point $P \in \partial\Omega$ has complex tangent space $\mathcal{T}_P(\partial\Omega)$. Note that $\partial\Omega$ has real dimension $2n-1$. If V is a small neighborhood of P in $\partial\Omega$, then we may write down a collection $L_1, \ldots, L_{n-1}$ of holomorphic vector fields on V that are linearly independent at each point of V. A commutator (or Poisson bracket) $[L_j, L_k]$ or $[L_j, \overline{L}_k]$ or $[\overline{L}_j, \overline{L}_k]$ is called a *second-order commutator*. If M is a $p^{\rm th}$-order commutator, then $[M, L_j]$ or $[M, \overline{L}_j]$ is called a $(p+1)^{\rm st}$-order commutator.

Definition 2 Let $\Omega \subseteq \mathbb{C}^n$ be a domain with C^∞ boundary. Let ρ be a smooth defining function for Ω. Let $P \in \partial\Omega$. Let m be a positive integer. We say that P has *commutator type* m if any commutator N of order $m-1$ or less satisfies

$$\langle N, \partial\rho \rangle = 0$$

but there is some commutator N' of order m that satisfies

$$\langle N', \partial\rho \rangle \neq 0\,.$$

The theorem of Bloom and Graham [BLG] says that, in $\mathbb{C}^2$, a point $P \in \partial\Omega$ is of geometric type m if and only if it is of commutator[3] type m. A brief proof of this statement appears in [KRA1, p. 469]. In higher dimensions this equivalence is still not fully understood, although there has been heartening recent progress by Fornæss and Lee [FOL].

John D'Angelo and David Catlin have demonstrated the importance of the concept of finite type, both for function theory and for the study of the $\overline{\partial}$-Neumann problem (see [DAN] and references therein). See also the work of Baouendi, Ebenfelt, and Rothschild [BER]. It is worthwhile to be able to understand points of finite type from a variety of different geometric points of view.

Our goal here is to understand the concept of finite type from the point of view of analytic discs, analogous to our understanding of pseudoconvexity in the last section. We continue to let "dist" denote Euclidean distance. We also let "H-dist" denote the Hausdorff distance on sets (see [KRP1], [KRP2]). The result we are about to present is certainly related to the work of Dwilewicz and Hill [DWH1], [DWH2]. These authors announce their results in all dimensions; but in the end they only prove them in dimension two. The results of the present paper are valid in all dimensions.

[3] There is an analogous result in dimension n for any n, but it necessitates a modified definition of "geometric type". The 2-dimensional result is implicit in the paper [KOH].

Proposition 4 *Fix a domain $\Omega \subseteq \mathbb{C}^n$ with smooth (that is, C^∞) boundary. Let $P \in \partial\Omega$. Fix an integer $m > 0$. If P has geometric type m then there is a sequence $\varphi_j : D \to \Omega$ of analytic discs satisfying*

(a) $\varphi_j(0) \to P$ *as* $j \to \infty$.

(b) $\text{diam}(\varphi_j(\overline{D})) \equiv \delta_j \to 0$ *as* $j \to \infty$.

(c) $\text{H-dist}(\varphi_j(\overline{D}), \partial\Omega) \leq \delta_j^m$.

Proof: Let $\varphi : D \to \mathbb{C}^n$ be an analytic disc that is tangent to $\partial\Omega$ at P to maximal order m as in line (3). Let ν be the unit outward normal vector to $\partial\Omega$ at P. Then the discs

$$\varphi_j = \varphi - \frac{1}{j}\nu$$

will satisfy the three conclusions of the proposition. In detail: For **(a)**, we get the result by inspection of the definition of φ_j. For property **(b)**, note that calculations in [KRA4] verify the result because the normal term in the Taylor expansion of the defining function is to the first power while the first tangential term is to the m^{th} power. The argument for **(c)** is similar.[4] □

Proposition 5 *Fix a domain $\Omega \subseteq \mathbb{C}^n$ with smooth (that is, C^∞) boundary. Let $P \in \partial\Omega$. Fix an integer $m > 0$. Assume that there is a sequence $\varphi_j : D \to \Omega$ of analytic discs satisfying*

(a) $\varphi_j(0) \to P$ *as* $j \to \infty$.

(b) $\text{diam}(\varphi_j(\overline{D})) \equiv \delta_j \to 0$ *as* $j \to \infty$.

(c) $\text{H-dist}(\varphi_j(\overline{D}), \partial\Omega) \leq \delta_j^m$.

Then P has geometric type at least m.

Proof: Consider a disc φ_j as given in the proposition. The fundamental theorem of calculus tells us that the tangent to the image disc at a point $\varphi_j(\zeta)$ must be tangent to the boundary at the corresponding projected point

[4] Of course φ_j will have to be replaced by $\varphi_j(c_j \cdot \zeta)$, with c_j a positive constant of size about $1/j^{1/m}$, so that the image disc lies entirely in Ω.

$\pi(\varphi_j(\zeta))$ (where $\pi : U \to \partial\Omega$ is Euclidean normal projection and U is a tubular neighborhood of $\partial\Omega$ as discussed earlier). In particular, there must be a point near the center of the disc at which the tangent is also tangent to the boundary. But then the normal translation of the disc will give an analytic disc tangent to the boundary at a point q_j. And the order of tangency must be at least m because of the condition **(c)**. These points q_j tend to the base point P by construction. Taking the limit in the definition (3) of finite type, one concludes that the point P has geometric type at least m. □

3 Other Geometric Conditions Involving Analytic Discs

Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Let $P \in \partial\Omega$. We now consider the following definition.

Definition 3 Suppose that $\psi : \overline{D} \to \overline{\Omega}$ is a closed analytic disc. Assume that whenever $\psi(\partial D) \subseteq \partial\Omega$ then $\psi(D) \cap \partial\Omega = \emptyset$. Then we say that every point of $\partial\Omega$ is *complex analytically extreme*.

This definition is analogous to the classical notion of "extreme point" from the theory of convex sets (see, for example, [VAL]). It is *not* the case that a domain satisfying the condition of this last definition must have the property that every boundary point is finite type. The example

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + 2e^{-1/|z_2|^2} < 1\}$$

illustrates the point. One might hope that every analytically extreme point is a peak point (see [KRA1, p. 141] for a thorough discussion); that would be a matter of considerable interest. This point is not completely understood at this time. However J. Yu's ideas about h-extendible points [YU1], [YU2] certainly shed light on the matter.

4 Some Examples, and Comparison with Harmonic Discs

EXAMPLE 4 Let

$$A = \{\zeta \in \mathbb{C} : 1/2 < |\zeta| < 2\}$$

and set

$$\Omega = A \times D\,.$$

Then Ω is a pseudoconvex domain in $\mathbb{C}^2$. It is a standard result—see [KRA1, p. 144]—that Ω may be exhausted by an increasing union of smoothly bounded, strongly pseudoconvex domains Ω_j. Thus we may choose j so large that

$$\mathrm{dist}(\partial\Omega_j, \partial\Omega) < 10^{-10}\,.$$

Now it is the case that the analytic disc

$$\begin{aligned} \psi : D &\rightarrow \Omega_j \\ \zeta &\mapsto (\zeta, 0) \end{aligned}$$

has the property that $\partial\psi$ lies in Ω_j while the entire disc does not.

This example does not contradict our characterization of pseudoconvexity with analytic discs (Proposition 1) because in that proposition we take the discs sufficiently small that they do not generate any nontrivial homology classes (see particularly the corresponding footnote).

EXAMPLE 5 Let

$$A = \{\zeta \in \mathbb{C} : 1/2 < |\zeta| < 2\}$$

and set

$$\widetilde{\Omega} = (A \times D) \cup \biggl(D(0,2) \times \{\zeta \in D : \mathrm{Re}\,\zeta < -3/4\} \biggr)\,.$$

The disc

$$\begin{aligned} \psi : D &\rightarrow \widetilde{\Omega} \\ \zeta &\mapsto (\zeta, 0) \end{aligned}$$

still has the property that $\partial\psi(D) \subseteq \widetilde{\Omega}$. Yet the full disc does not lie in $\widetilde{\Omega}$.

However, note that this Ω is *not* pseudoconvex. Indeed the smallest pseudoconvex domain that contains Ω is $D(0,2) \times D$. Thus, even though the disc ψ has boundary curve that is homotopic to a point, the example is insignificant because the domain is not pseudoconvex.

It is natural to wonder whether analytic discs are the optimal device to use to measure pseudoconvexity. Perhaps harmonic discs—which we use in effect to recognize plurisubharmonic functions—are more appropriate. They would certainly be more flexible. Here by a *harmonic disc* we mean the following. Let $\eta : \{e^{i\theta} : 0 \leq \theta \leq 2\pi\} \to \mathbb{C}^n$ be a continuous function with $\eta(e^{i0}) = \eta(e^{i2\pi})$. Now solve the Dirichlet problem with boundary data η to obtain a harmonic function

$$u : D \to \mathbb{C}^n$$

that is continuous up to the boundary. Is there a characterization of pseudoconvex domain using harmonic discs that is analogous to Proposition 1?

Certainly we may note that if **(i)** whenever the boundary of an analytic disc is in Ω then the corresponding harmonic disc is in Ω, then **(ii)** certainly the same is true for holomorphic discs. So, by what we have already proved, the domain must be pseudoconvex. What is false is the converse, as the next example shows.

EXAMPLE 6 Let

$$E = \{\zeta \in \mathbb{C} : 1 < \max\{|\mathrm{Re}\,\zeta|, |\mathrm{Im}\,\zeta| < 2\} .$$

This is a large square box with a smaller square box removed. Let

$$\Omega = E \times D .$$

Of course Ω is pseudoconvex.

Let $\eta : [0, 1] \to \Omega$ be a closed curve such that

- $\eta(t) = (\eta_1(t), 0)$;
- the image of η lies within $\epsilon > 0$ of the boundary for some small ϵ;
- η_1 traces entirely along two edges of the square $\{\max\{|\mathrm{Re}\,\zeta|, |\mathrm{Im}\,\zeta|\} = 1 + \epsilon/2\}$.

Then harmonic measure (see [GAM] or [KRA3]) shows that the harmonic disc with boundary curve η will have points lying outside Ω.

5 Introduction to the Kobayashi Metric

The Kobayashi or Kobayashi/Royden metric $F_K^\Omega(z,\xi)$, and its companion the Carathéodory metric $F_C^\Omega(z,\xi)$, has, in the past 40 years, proved to be an important tool in the study of function-theoretic and geometric properties of complex analytic objects—see [KOB1], [KOB2], [KOB3], for example. One remarkable feature of this tool is that quite a lot of mileage can be had just by exploiting formal properties of the metric (e.g., the distance non-increasing property under holomorphic mappings). See [JAP], [EIS], [KRA2]. But more profound applications of these ideas require hard analytic properties of the metric. One of the first, and most profound, instances of this type of work is [GRA]. Another is [LEM].

Our goal now is to continue some of the basic development of analytic facts about the Carathéodory and Kobayashi metrics. Our focus in fact is on regularity of the metric. It is well known (see [JAP, pp. 98–99]) that the infinitesimal Kobayashi metric is *always* upper semicontinuous. The reference [JAP] goes on to note that, on a taut domain,[5] the infinitesimal metric is in fact *continuous*. Continuity of the Carathéodory metric was studied, for example, in [GRK1].

But one would like to know more. For various regularity results, and applications in function theory, it is useful to know that the infinitesimal metrics are *Lipschitz* as a function of the two arguments. In the present paper we prove such a result (in the case of the Kobayashi metric) when the domain of study Ω is smoothly bounded and strongly Levi pseudoconvex; we also prove such a result for the Carathéodory metric.

All necessary definitions of key concepts will be provided below.

6 Fundamental Concepts Concerning the Kobayashi Metric

In the discussion that follows we let D denote the unit disc in $\mathbb{C}$.

A *domain* $\Omega \subseteq \mathbb{C}^n$ is a connected, open set. It is common to let $U_1(U_2)$ denote the collection of holomorphic mappings from U_2 to U_1. The *infinitesimal*

[5]Here a domain Ω is taut, as originally defined by H. H. Wu in [WU], if the family of holomorphic mappings of the disc D into Ω is normal. Kerzman [KER] has shown that a C^2 pseudoconvex domain is taut.

Kobayashi metric on Ω is defined, for $P \in \Omega$ and $\xi \in \mathbb{C}^n$, to be

$$\begin{aligned}
&F_K^\Omega(P,\xi) \\
&\quad \equiv \ \inf\{\alpha : \alpha > 0 \text{ and } \exists f \in \Omega(D) \text{ with } f(0) = P, f'(0) = \xi/\alpha\} \\
&\quad = \ \inf\left\{\frac{|\xi|}{|f'(0)|} : f \in \Omega(D), f(0) = P, f'(0) \text{ is a positive, constant multiple of } \xi\right\} \\
&\quad = \ \frac{|\xi|}{\sup\{|(f'(0))| : f \in \Omega(D), f(0) = P, f'(0) \text{ is a positive, constant multiple of } \xi\}} \, .
\end{aligned}$$

Here $|\xi|$ denotes the standard Euclidean length of the vector ξ.

It is frequently convenient to think of ξ as an element of the tangent space to Ω at z; this notion will have no bearing on the present discussion. There is also an integrated form of the Kobayashi metric (see [KRA1, p. 437]). It will play only a tacit role in the present paper.

It is well known that, if $\Phi : \Omega_1 \to \Omega_2$ is a holomorphic (not necessarily biholomorphic) mapping, then

$$F_K^{\Omega_1}(z,\xi) \geq F_K^{\Omega_2}(\Phi(z), \Phi_*\xi) \, .$$

Here $\Phi_*\xi$ is the standard push-forward of the vector ξ (see [FED]). It follows immediately that, in case Φ is *biholomorphic*, then Φ induces an isometry of Kobayashi metrics.

We shall do our work in this paper on *Levi pseudoconvex domains*. This terminology and related concepts was considered in some detail in the first part of the paper. See also [KRA1, p. 127].

7 The Main Result

The principal result about the Kobayashi metric is as follows.

Theorem 6 *Assume that Ω is strongly pseudoconvex with C^6 boundary. Let $K \subseteq \Omega$ and $L \subseteq \mathbb{C}^n$ be compact sets. There is a constant $C = C_{K,L} > 0$ such that, if $z, z' \in K$ and $\xi, \xi' \in L$, then*

$$\left|F_K^\Omega(z,\xi) - F_K^\Omega(z',\xi')\right| \leq C_{K,L} \cdot \left[\sqrt{|z - z'|^2 + |\xi - \xi'|^2}\right]^{2/3} .$$

8 Proof of the Main Result

This section is dedicated to the proof of Theorem 6. We begin with two lemmas.

Lemma 7 *Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded, strongly pseudoconvex domain. Let $P \in \Omega$ be sufficiently near the boundary, and let ξ be a* complex transversal *direction at P. We may assume that ξ is a Euclidean unit vector. Fix a compact subset $K \subset \Omega$. There is a universal constant $c > 0$ with the following property:*

Let φ be a Kobayashi extremal disc for the point P in the direction ξ. If $\epsilon > 0$ is small and $\zeta \in D$ has distance ϵ from the boundary of D, then

$$dist_{\mathrm{Eucl}}(\varphi(\zeta), \partial\Omega) \approx c \cdot \epsilon \,.$$

If instead ξ is a Euclidean unit vector that is complex tangential *at P, then we have the estimate*

$$dist_{\mathrm{Eucl}}(\varphi(\zeta), \partial\Omega) \approx c \cdot \epsilon^{3/2} \,.$$

The converses of both these statements are true as well.

Proof: The first result follows from the main results of [HUA1, Theorem 1, p. 284, Corollary 1, p. 284] and [HUA2, Theorem 2, p. 400]. The main fact is that this extremal mapping φ is totally geodesic provided that P is close enough to the boundary and ξ is complex transversal. The results of Huang require C^3 boundary.

The second result follows from estimates in [BUK, Prop. 4.3, p. 666]. The main idea is that, as an elementary calculation shows, a point that is distance ϵ from the boundary inside a *tangential* extremal disc (which is totally geodesic by results of [BUK] as indicated) actually has normal Euclidean distance about $\epsilon^{3/2}$ from the boundary of Ω. Note that the results of [BUK] require the domain to have C^6 boundary.

See also [FU] for information about extremal properties of the Kobayashi metric and discs. □

Lemma 8 *Let U be a smoothly bounded domain in $\mathbb{C}$. There are positive constants C', C'' with the following property. Consider the differential equation*

$$\frac{\partial}{\partial\overline{\zeta}} u(\zeta) = f(\zeta)\, d\zeta \tag{5}$$

on U. Suppose that f is C^1, and that there is a constant $C_1 > 0$ such that $|f| \leq C_1$, and that $|\nabla f| \leq C_1$. Then there is a solution u for (5) that satisfies

$$|u| \leq C_1 \cdot C' \qquad \text{and} \qquad |\nabla u| \leq C_1 \cdot C'' .$$

Proof: This result may be found in [KRA1, p. 411 ff.] and references therein. □

Proof of the Main Theorem:

We separate the case of perturbation of the base point P from the case of perturbation of the tangent vector ξ.

Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain with C^6 boundary, $P \in \Omega$ a point, and $\xi \in \mathbb{C}^n$ a tangent vector that is unit in the Euclidean metric.

We have observed that the domain Ω is taut. So a simple normal families argument (see [JAP]) shows that there is a mapping $\varphi : D \to \Omega$ with $\varphi(0) = P$ and $\varphi'(0)$ parallel to ξ so that

$$\begin{aligned} \frac{|\xi|}{|\varphi'(0)|} &= F_K^\Omega(z,\xi) \\ &= \inf\left\{ \frac{|\xi|}{|f'(0)|} : f \in \Omega(D), f(0) = P, \right. \\ &\qquad \left. f'(0) \text{ is a positive constant multiple of } \xi \right\} . \end{aligned}$$

Now let $\eta : D \to \mathbb{R}^+$ be a C^∞ cutoff function with these properties:

(a) $0 \leq \eta \leq 1$;

(b) $\eta \equiv 1$ on $D(0, 1/4)$;

(c) $\eta(\zeta) = 0$ for $|\zeta| > 1/2$.

Certainly we may say that $|\nabla \eta| \leq C_1$, $|\nabla^2 \eta| \leq C_1$ for some $C_1 > 0$.

Define $k_0 = [48(C' + C'' + C_1 + c + 10)]^2$, where C', C'' are the constants from the estimates for the $\overline{\partial}$ problem (see Lemma 8), C_1 is the constant from the estimates on $|\nabla \eta|$, and c comes from Lemma 7. Let μ be any Euclidean unit vector in $\mathbb{C}^n$. Let $\epsilon > 0$ be small. Then let

$$h(\zeta) = \eta(\zeta) \cdot [\varphi(\zeta) + (\epsilon^{3/2}/k_0) \cdot \mu] + [1 - \eta(\zeta)] \cdot \varphi(\zeta) + \sqrt{\epsilon^{3/2}/k_0} \cdot \zeta^2 \cdot \chi(\zeta) . \quad (6)$$

There are three terms on the righthand side of the definition of h. The first two of these should be thought of as a small perturbation of the extremal mapping φ. The third is a correction term which we hope to choose (using the $\overline{\partial}$ problem) so as to make h holomorphic.

Now we have

$$0 = \overline{\partial} h = \overline{\partial}\eta \cdot [\varphi + (\epsilon^{3/2}/k_0) \cdot \mu] - \overline{\partial}\eta \cdot \varphi + \sqrt{\epsilon^{3/2}/k_0} \cdot \zeta^2 \cdot \overline{\partial}\chi \, .$$

Thus we must solve the equation

$$\overline{\partial}\chi = \frac{\sqrt{k_0}}{\sqrt{\epsilon^{3/2}} \cdot \zeta^2}\left[-\overline{\partial}\eta[\varphi + (\epsilon^{3/2}/k_0)\mu] + \overline{\partial}\eta \cdot \varphi \right] = -\frac{\overline{\partial}\eta \cdot \sqrt{\epsilon^{3/2}} \cdot \mu}{\sqrt{k_0} \cdot \zeta^2} \, . \qquad (7)$$

Of course $\overline{\partial}\eta$ vanishes in a neighborhood of $\zeta = 0$ so that the righthand side of (6) is well defined and smooth.

If we take $|\nabla \eta| \leq C$ and $|\nabla^2 \eta| \leq C$ then of course the righthand side of this last equation, together with its first derivatives, is bounded by $C\sqrt{\epsilon^{3/2}}/[\sqrt{k_0} \cdot (1/4)^2]$. It is also $\overline{\partial}$-closed. Thus, by Lemma 8, the equation (7) has a bounded solution χ with bound $C_1 \cdot C' \sqrt{\epsilon^{3/2}}/[\sqrt{k_0} \cdot (1/4)^2] \leq \sqrt{\epsilon^{3/2}}/[4(c+1)]$; also that solution has bounded gradient—smaller than $C_1 \cdot C'' \sqrt{\epsilon^{3/2}}/[\sqrt{k_0} \cdot (1/4)^2] \leq \sqrt{\epsilon^{3/2}}/[4(c+1)]$. Thus if we define $\widetilde{h}(\zeta) = h((1-\epsilon)\zeta)$, then (by Lemma 7) $\widetilde{h}$ maps D to Ω. Thus $\widetilde{h}$ is a good *candidate disc* for the Kobayashi metric for the point $\widetilde{h}(0) = P + (\epsilon^{3/2}/k_0) \cdot \mu = \widetilde{P}$. Note that $\widetilde{h}$ is not necessarily extremal.

Putting the function χ into (6), we can be sure (because the third term on the righthand side of (6) is plainly much smaller than the other two—after all, η is bounded from 0 on a large set and $\varphi'(0)$ is bounded from 0) that χ does not simply cancel the first two terms in the definition of h. And we now know that h is holomorphic.

We now compare the infinitesimal Kobayashi metric at the base point P in the direction ξ with the metric at the base point $\widetilde{P}$ in the direction ξ. What we must examine is

$$\frac{|\xi|}{|\varphi'(0)|} - \frac{|\xi|}{|\widetilde{h}'(0)|} = \frac{|\xi| \cdot [|\widetilde{h}'(0)| - |\varphi'(0)|]}{|\varphi'(0)| \cdot |\widetilde{h}'(0)|} \, .$$

Of course the denominator is bounded from zero and $|\xi| = 1$. We see that

$$\begin{aligned} \big| |\widetilde{h}'(0)| - |\varphi'(0)| \big| &\leq \big| \widetilde{h}'(0) - \varphi'(0) \big| \\ &= \big| (1-\epsilon)h'(0) - \varphi'(0) \big| \, . \end{aligned}$$

And now we use the original definition of h to see that this last difference is

$$(1-\epsilon)\varphi'(0) - \varphi'(0)\,.$$

Thus we have an error term of the form $-\epsilon\varphi'(0)$.

In detail, what we have proved is that, for any extremal disc φ at (P,ξ), there is a "nearby" candidate disc (not necessarily extremal) $\widetilde{h}$ for $(\widetilde{P},\xi)$ that satisfies favorable estimates. Thus

$$\frac{|\xi|}{|\widetilde{h}'(0)|} - c\cdot\epsilon \leq \frac{|\xi|}{|\varphi'(0)|}\,, \tag{8}$$

where ϵ is small and depends on ϵ, ϵ, and ϵ in obvious ways. In other words,

$$\frac{|\xi|}{|\widetilde{h}'(0)|} - c\cdot\epsilon \leq F_K^{\Omega}(P,\xi)\,. \tag{9}$$

A similar argument, reversing the roles of P and $\widetilde{P}$, shows that

$$\frac{|\xi|}{\left|\widetilde{\widetilde{h}}'(0)\right|} - c\cdot\epsilon \leq F_K^{\Omega}(\widetilde{P},\xi) \tag{10}$$

for some other candidate mapping $\widetilde{\widetilde{h}}$ at P for the direction ξ, and for some small positive number ϵ.

But (10) certainly says that

$$F_K^{\Omega}(\widetilde{P},\xi) \leq \frac{|\xi|}{|\widetilde{h}'(0)|} \leq F_K^{\Omega}(P,\xi) + c\cdot\epsilon$$

and (10) says that

$$F_K^{\Omega}(P,\xi) \leq \frac{|\xi|}{\left|\widetilde{\widetilde{h}}'(0)\right|} \leq F_K^{\Omega}(\widetilde{P},\xi) + c\cdot\epsilon\,.$$

We thus conclude that

$$|F_K^{\Omega}(P,\xi) - F_K^{\Omega}(\widetilde{P},\xi)| \leq c\cdot\epsilon + c\cdot\epsilon\,.$$

But note that $|P-\widetilde{P}| = \epsilon^{3/2}/k_0$. So we may conclude that F_K^{Ω} is Lipschitz of order $2/3$. $\square$

Next one must examine variation in the tangent vector ξ. But the argument is substantially the same. We may treat rotation of the tangent vector ξ and dilation of the tangent vector ξ separately.

For the case of rotation, we let Ξ be a unitary rotation on $\mathbb{C}^n$ that is near to the identity and we set

$$h(\zeta) = \eta(\zeta) \cdot \Xi[\varphi(\zeta)] + [1 - \eta(\Xi\zeta)] \cdot \varphi(\zeta) + \sqrt{\epsilon/k_0} \cdot \zeta^2 \cdot \chi(\zeta) \,. \tag{11}$$

and then argues precisely as above. One should note here that $h'(0) = \Xi(\xi)$, just as we wish.

For the case of dilation, we let $\epsilon^* > 0$ and define

$$h(\zeta) = \eta(\zeta) \cdot (1 + \epsilon^*)\varphi(\zeta) + [1 - \eta(\zeta)] \cdot \varphi(\zeta) + \sqrt{\epsilon} \cdot \zeta^2 \cdot \chi(\zeta) \,. \tag{12}$$

and then argues just as before. One should note here that $h'(0) = (1 + \epsilon^*)\xi$, just as we wish.

It may be noted that our argument applies uniformly over a neighborhood of P (the size of the neighborhood depending on ϵ), so that a standard compactness argument shows that our estimates obtain uniformly over compact sets.

That completes our argument, and proves the theorem. □

9 The Carathéodory Metric

As a complement to the result of the preceding two sections, we now prove a result about the Carathéodory metric. A form of the result presented here appears in Proposition 2.5.1 of [JAP]. We include it here for completeness.

We begin with a quick review of that metric.

Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $P \in \Omega$, and $\xi \in \mathbb{C}^n$ a vector. Then we define the *infinitesimal Carathéodory metric* at P in the direction ξ to be

$$F_C(P, \xi) = \sup_{\substack{f \in D(\Omega) \\ f(P)=0}} |f_*(P)\xi| \equiv \sup_{\substack{f \in D(\Omega) \\ f(P)=0}} \left| \sum_{j=1}^n \frac{\partial f}{\partial z_j}(P) \cdot \xi_j \right| .$$

Now we have

Theorem 9 *Let $\Omega \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with C^3 boundary. Let $K \subseteq \Omega$ and $L \subseteq \mathbb{C}^n$ be compact sets. There is a constant $C = C_{K,L} > 0$ such that, if $z, z' \in K$ and $\xi, \xi' \in L$, then*

$$\left|F_C^\Omega(z,\xi) - F_C^\Omega(z',\xi')\right| \le C_{K,L} \cdot \sqrt{|z-z'|^2 + |\xi - \xi'|^2}\,.$$

Proof: Let $P \in \Omega$ be a fixed point and $\xi \in \mathbb{C}^n$ a fixed vector. Let $\varphi : \Omega \to D$ be a candidate mapping for the infinitesimal Carathéodory metric at P in the direction ξ. Let $\epsilon > 0$.

Now let $P' \in \Omega$ be a point that is near to P. Let $\eta \in C_c^\infty(\mathbb{C}^n)$ be a cutoff function that is equal to 1 near P (so that it is identically 1 in a neighborhood of P and also in a neighborhood of P'). Define

$$h(z) = \eta(z) \cdot \varphi(P + (z - P')) + (1 - \eta(z)) \cdot \varphi(z) + \chi(z)\,.$$

We think of h as a small perturbation of the extremal mapping φ. Notice that $h(P') = 0 + \chi(P')$. We want to select χ, using the theory of the $\overline{\partial}$ problem, so that h is holomorphic.

We have

$$\begin{aligned} \overline{\partial}\chi(z) &= \overline{\partial}\eta(z) \cdot \varphi(z) - \overline{\partial}\eta(z) \cdot \varphi(P + (z - P')) \\ &= \overline{\partial}\eta(z) \cdot (\varphi(z) - \varphi(P + (z - P')))\,. \end{aligned} \tag{12}$$

The righthand side is of course $\overline{\partial}$-closed.

Notice that the righthand side of (12) is small (less than a universal constant C_1 times ϵ) in the uniform topology provided only that P' is sufficiently close to P. In fact the same reasoning shows that it is small in the C^1 topology. Thus the theory of the $\overline{\partial}$ problem on strongly pseudoconvex domains (see [KRA1, p. 411 ff.], for instance) tells us that we may choose χ to satisfy (12) and so that χ is C^1 small (i.e., bounded by a universal constant times ϵ) provided only that P' is sufficiently close to P. Thus h is holomorphic and close to φ in the C^1 topology. Consequently, for P' sufficiently close to P, we have

$$\sup_{z \in K} |h(z) - \varphi(z)| < \epsilon\,.$$

Now we set

$$\widetilde{h}(z) = \eta(z)\varphi(P + (z - P')) + (1 - \eta(z))\varphi(z) + \chi(z) - \chi(P')$$

and

$$\widetilde{\varphi}(z) = \frac{1}{1 + (C_1 + 1) \cdot \epsilon} \cdot \widetilde{h} \, .$$

We see that $\widetilde{\varphi}$ is a holomorphic mapping from Ω to D, it takes the value 0 at P', and it is close to φ in the C^1 topology. We may also conclude that the existence of the mapping $\widetilde{\varphi}$ shows that the Carathéodory metric at P' in the direction $\xi' = \xi$ is C^1-close to the metric at P in the direction ξ. To wit,

$$|\varphi_*(P)\xi| \qquad \text{and} \qquad |\widetilde{\varphi}_*(P)\xi|$$

are close.

The same argument holds in reverse if we choose P' as the base point at P as the perturbed point. We may conclude therefore that the infinitesimal Carathéodory metric varies in the Lipschitz topology when the base point is perturbed.

A similar, but even easier, argument applies (just as in our discussion of the Kobayashi metric) when the tangent vector ξ is perturbed. That completes our argument. $\square$

10 Concluding Remarks

We have presented some new ways to think about the classical concept of pseudoconvexity. Of course our presentation has roots in the most fundamental ideas of the subject, but the formulations are new.

It is a natural next question to consider regularity of the infinitesimal invariant metrics. Our motivation was the study of extremal discs for the Kobayashi metric in the sense of Lempert [LEM]. But there are many contexts in which estimates of this kind may prove useful.

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