# Regularity of the Kobayashi and Carathéodory Metrics on Levi Pseudoconvex Domains

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## 0 Introduction

The Kobayashi or Kobayashi/Royden metric $F_K^\Omega(z,\xi)$, and its companion the Carathéodory metric $F_C^\Omega(z,\xi)$, has, in the past 40 years, proved to be an important tool in the study of function-theoretic and geometric properties of complex analytic objects—see [KOB1], [KOB2], [KOB3], for example. One remarkable feature of this tool is that quite a lot of mileage can be had just by exploiting formal properties of the metric (e.g., the distance non-increasing property under holomorphic mappings). See [JAP], [EIS], [KRA2]. But more profound applications of these ideas require hard analytic properties of the metric. One of the first, and most profound, instances of this type of work is [GRA]. Another is [LEM].

Our goal in this paper is to continue some of the basic development of analytic facts about the Carathéodory and Kobayashi metrics. Our focus in fact is on regularity of the metric. It is well known (see [JAP, pp. 98–99) that the infinitesimal Kobayashi metric is *always* upper semicontinuous. The reference [JAP] goes on to note that, on a taut domain, the infinitesimal metric is in fact *continuous*. Continuity of the Carathéodory metric was studied, for example, in [GRK].

But one would like to know more. For various regularity results, and applications in function theory, it is useful to know that the infinitesimal

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metrics are *Lipschitz* as a function of the two arguments. In the present paper we prove such a result (in the case of the Kobayashi metric) when the domain of study $\Omega$ is smoothly bounded and Levi pseudoconvex; we also prove such a result (for the Carathéodory metric) when the domain in question is smoothly bounded and *strongly* pseudoconvex.

All necessary definitions of key concepts will be provided below.

# 1 Fundamental Concepts

A *domain* $\Omega \subseteq \mathbb{C}^n$ is a connected, open set. We let $D \subseteq \mathbb{C}$ denote the unit disc. It is common to let $(U_1, U_2)$ denote the collection of holomorphic mappings from $U_2$ to $U_1$. The *infinitesimal Kobayashi metric* on $\Omega$ is defined, for $z \in \Omega$ and $\xi \in \mathbb{C}^n$, to be

$$\begin{aligned} F_K^\Omega(z,\xi) &\equiv \inf\{\alpha : \alpha > 0 \text{ and } \exists f \in \Omega(B) \text{ with } f(0) = z, (f'(0))(e_1) = \xi/\alpha\} \\ &= \inf\left\{\frac{|\xi|}{|(f'(0))(e_1)|} : f \in \Omega(B), (f'(0))(e_1) \text{ is a} \right. \\ &\qquad\qquad \text{constant multiple of } \xi\} \\ &= \frac{|\xi|}{\sup\{|(f'(0))(e_1)| : f \in \Omega(B), (f'(0))(e_1) \text{ is a constant multiple of } \xi\}} . \end{aligned}$$

Here $e_1$ is simply the unit tangent vector $(1, 0, \dots, 0) = (1 + i0, 0 + i0, \dots, 0 + i0)$ in the unit disc. Also $|\xi|$ denotes the standard Euclidean length of the vector $\xi$.

It is frequently convenient to think of $\xi$ as an element of the tangent space to $\Omega$ at $z$; this notion will have no bearing on the present discussion. There is also an integrated form of the Kobayashi metric (see [KRA1]). It will play only a tacit role in the present paper.

It is well known that, if $\Phi : \Omega_1 \to \Omega_2$ is a holomorphic (not necessarily biholomorphic) mapping, then

$$F_K^{\Omega_1}(z,\xi) \geq F_K^{\Omega_2}(\Phi(z), \Phi_*\xi) .$$

Here $\Phi_*\xi$ is the standard push-forward of the vector $\xi$ (see [FED]). It follows immediately that, in case $\Phi$ is *biholomorphic*, then $\Phi$ induces an isometry of Kobayashi metrics.

We shall do our work in this paper on *Levi pseudoconvex domains*. This terminology is standard in the subject, and we refer the reader to [KRA1] for the details.

## 2 The Main Result

Fix a Levi pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$ with $C^2$ boundary. This means that the Levi form at each boundary point is positive semi-definite (see [KRA1] for the details). Then we have:

**Theorem 1** *Let $K \subseteq \Omega$ and $L \subseteq \mathbb{C}^n$ be compact sets. There is a constant $C = C_{K,L} > 0$ such that, if $z, z' \in K$ and $\xi, \xi' \in L$, then*

$$\left| F_K^\Omega(z, \xi) - F_K^\Omega(z', \xi') \right| \leq C \cdot \sqrt{|z - z'|^2 + |\xi - \xi'|^2} \, .$$

## 3 Proof of the Main Result

This section is dedicated to the proof of Theorem 1. We begin with a lemma.

**Lemma 2** *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, Levi pseudoconvex domain with $C^2$ boundary. Let $K$ be a relatively compact subset of $\Omega$. Fix a number $\delta > 0$. Then there exists a number $\epsilon > 0$ such that, if $P \in K$, $\xi \in \mathbb{C}^n$ is an arbitrary Euclidean unit vector, $\varphi : D \to \Omega$ is Kobayashi extremal for the base point $P$ in the direction $\xi$, and $|\zeta| \leq 1 - \delta$, then*

$$dist(\varphi(\zeta), \partial\Omega) > \epsilon \, .$$

**Proof:** Suppose not. Then, for some fixed $\delta > 0$, no corresponding number $\epsilon > 0$ exists.

This means that, for $j = 1, 2, \ldots$ there are $\zeta_j \in D$ with $|\zeta_j| \leq 1 - \delta$, $P_j \in K$, unit vectors $\xi_j \in \mathbb{C}^n$, and Kobayashi extremal maps $\varphi_j : D \to \Omega$ for $(P_j, \xi_j)$ such that

$$\text{dist}(\varphi_j(\zeta_j), \partial\Omega) \leq 2^{-j} \, . \qquad (*)$$

Notice that the family of holomorphic functions $\{\varphi_j\}$ on the unit disc $D$ with values in $\Omega$ is uniformly bounded. Thus Montel's theorem applies and we may extract a subsequence $\{\varphi_{j_k}\}$ which converges uniformly on compact subsets of $D$ to a limit mapping $\varphi_0 : D \to \Omega$. We use here the fact that $\Omega$ is taut (see [WU][, [KER]). Since the unit sphere in $\mathbb{C}^n$ is compact, we may also (by a simple diagonalization argument) suppose that $\xi_{j_k}$ converges to a limit vector $\xi_0$. Next, since $K$ is compact, we may assume that the $P_{j_k}$ converge to a limit point $P_0$. Finally, we may assume that $\zeta_{j_k}$ converge to a limit point $\zeta_0$ with $|\zeta_0| \leq 1 - \delta$.

But it follows then from $(*)$ that $\varphi_0(\zeta_0) \in \partial\Omega$. This is of course impossible, by elementary properties of smoothly bounded, pseudoconvex domains, unless $\varphi_0$ is constant (this is the Kontinuitätssatz—see [KRA1]). But it is not (since $\varphi_0'(0) \neq 0$). So we have a contradiction. $\square$

We have observed that the domain $\Omega$ is taut. So a simple normal families argument (see [JAP]) shows that there is a mapping $\varphi : D \to \Omega$ with $\varphi(0) = z$ and $\varphi'(0)$ parallel to $\xi$ so that

$$\begin{aligned} \frac{|\xi|}{|\varphi'(0)|} &= \frac{|\xi|}{|(\varphi'(0))(e_1)|} \\ &= F_K^\Omega(z, \xi) \\ &= \inf \left\{ \frac{|\xi|}{|(f'(0))(e_1)|} : f \in \Omega(B), (f'(0))(e_1) \text{ is a constant multiple of } \xi \right\}. \end{aligned}$$

We prove the desired inequality in two parts: namely, we examine variation in the base point $z$ and then we separately examine variation in the tangent vector $\xi$. Let us now inspect the first of these.

Fix a point $P \in \Omega$ and a unit vector $\xi \in \mathbb{C}^n$. Let $\varphi : D \to \Omega$ be extremal for $(P, \xi)$. Let $\mu$ be any unit vector in $\mathbb{C}^n$ and $\widetilde{\epsilon} > 0$ be small.

Now let $\eta : D \to \mathbb{R}^+$ be a $C^\infty$ cutoff function with these properties:

**(a)** $0 \leq \eta \leq 1$;

**(b)** $\eta \equiv 1$ in a neighborhood of $0 \in D$;

**(c)** $\eta(\zeta) = 0$ for $|\zeta| > 1/2$.

Define

$$h(\zeta) = \eta(\zeta) \cdot [\varphi(\zeta) + \widetilde{\epsilon}\mu] + [1 - \eta(\zeta)] \cdot \varphi(\zeta) + \zeta^2 \cdot \chi(\zeta) . \qquad (**)$$

There are three terms on the righthand side of the definition of $h$. The first two of these should be thought of as a small perturbation of the extremal mapping $\varphi$. The third is a correction term which we hope to choose (using the $\overline{\partial}$ problem) so as to make $h$ holomorphic.

Now we have

$$0 = \overline{\partial} h(\zeta) = \overline{\partial}\eta \cdot [\varphi(\zeta) + \widetilde{\epsilon}\mu] - \overline{\partial}\eta \cdot \varphi(\zeta) + \zeta^2 \cdot \overline{\partial}\chi .$$

Thus we must solve the equation

$$\overline{\partial}\chi = \frac{1}{\zeta^2}\left[-\overline{\partial}\eta[\varphi(\zeta)+\widetilde{\epsilon}\mu] + \overline{\partial}\eta(\zeta)\cdot\varphi(\zeta)\right] = -\frac{\overline{\partial}\eta\cdot\widetilde{\epsilon}\mu}{\zeta}\,. \qquad (\star)$$

If we take $|\nabla\eta| \le C$ and $|\nabla^2\eta| \le C$ then of course the righthand side of this last equation, together with its first derivatives, is bounded by $C\widetilde{\epsilon}$. It is also $\overline{\partial}$-closed. Thus, by well-known estimates (see [KRA1] and references therein) the equation $(\star)$ has a bounded solution $\chi$ with bound $C'\cdot\widetilde{\epsilon}$; also that solution has bounded gradient. Putting this function into $(**)$, we can be sure that $\chi$ does not simply cancel the first two terms in the definition of $h$. And we now know that $h$ is holomorphic.

Most significantly, if $\widetilde{\epsilon}$ is chosen to be small enough, then the lemma will guarantee that $h : D \to \Omega$. We think of $h$ as a candidate holomorphic mapping for the base point $\widetilde{P} \equiv P + \epsilon\mu$ and tangent vector $\widetilde{\xi} = \xi$.

In summary, we have shown that if we perturb the pair $(P,\xi)$ to the new pair $(\widetilde{P},\widetilde{\xi})$, then the extremal map $\varphi$ is perturbed to a nearby map $\widetilde{\varphi} = h$. The argument works just as well in reverse to show that any extremal map for $(\widetilde{P},\widetilde{\xi})$ perturbs to a nearby mapping for $(P,\xi)$. The upshot is that a small perturbation of the base point gives a small perturbation of the extremal map; and inspection of the argument shows that the variation is Lipschitz.

Next one must examine variation in the tangent vector $\xi$. But the argument is substantially the same. In the definition of $h$, one replaces (in the first term) the expression

$$\varphi(\zeta) + \widetilde{\epsilon}\mu$$

with either

$$\varphi(\zeta)\cdot\alpha\,,$$

some unit vector $\alpha$ (with $|\alpha - 1|$ small), for a rotation through angle $\alpha$ of the tangent vector, or with

$$\varphi(\zeta)\cdot(1+\epsilon^*)$$

for a dilation of the tangent vector. The error term is still $\zeta^2\cdot\chi$ (so that the tangent vector is unchanged at the base point), and the rest of the argument remains unchanged. And then one solves a $\overline{\partial}$ problem to obtain a perturbed mapping $\widetilde{\varphi}$.

That completes our argument, and proves the proposition.

# 4 The Carathéodory Metric

As a complement to the result of the preceding two sections, we now prove a result about the Carathéodory metric. A form of the result presented here appears in Proposition 2.5.1 of [JAP]. We include it here for completeness.

We begin with a quick review of that metric.

Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $P \in \Omega$, and $\xi \in \mathbb{C}^n$ a vector. Then we define the *infinitesimal Carathéodory metric* at $P$ in the direction $\xi$ to be

$$F_C(z,\xi) = \sup_{\substack{f \in D(\Omega) \\ f(z)=0}} |f_*(z)\xi| \equiv \sup_{\substack{f \in D(\Omega) \\ f(z)=0}} \left| \sum_{j=1}^n \frac{\partial f}{\partial z_j}(z) \cdot \xi_j \right| .$$

Now we have

**Theorem 3** *Let $\Omega \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with $C^2$ boundary. Then the infinitesimal Carathéodory metric on $\Omega$ is Lipschitz in the sense that if if $z, z' \in \Omega$ and $\xi, \xi' \in \mathbb{C}^n$, then*

$$\left| F_C^\Omega(z,\xi) - F_C^\Omega(z',\xi') \right| \le C \cdot \sqrt{|z-z'|^2 + |\xi - \xi'|^2} .$$

**Proof:** Let $P \in \Omega$ be a fixed point and $\xi \in \mathbb{C}^n$ a fixed vector. Let $\varphi : \Omega \to D$ be a candidate mapping for the infinitesimal Carathéodory metric at $P$ in the direction $\xi$. Let $\epsilon > 0$.

Now let $P' \in \Omega$ be a point that is near to $P$. Let $\eta \in C_c^\infty(\mathbb{C}^n)$ be a cutoff function that is equal to 1 near $P$ (so that it is identically 1 in a neighborhood of $P$ and also in a a neighborhood of $P'$). Define

$$h(z) = \eta(z) \cdot \varphi(P + (z - P')) + (1 - \eta(z)) \cdot \varphi(z) + \chi(z) .$$

We think of $h$ as a small perturbation of the extremal mapping $\varphi$. Notice that $h(P') = 0 + \chi(P')$. We want to select $\chi$, using the theory of the $\overline{\partial}$ problem, so that $h$ is holomorphic.

We have

$$\overline{\partial}\chi(z) = \overline{\partial}\eta(z) \cdot \varphi(z) - \overline{\partial}\eta(z) \cdot \varphi(P + (z - P')) = \overline{\partial}\eta(z) \cdot (\varphi(z) - \varphi(P + (z - P'))) . \qquad (*)$$

The righthand side is of course $\overline{\partial}$-closed.

Notice that the righthand side of $(*)$ is small in the uniform topology provided only that $P'$ is sufficiently close to $P$. In fact the same reasoning shows that it is small in the $C^1$ topology. Thus the theory of the $\overline{\partial}$ problem on strongly pseudoconvex domains (see [KRA1], for instance) tells us that we may choose $\chi$ to satisfy $(*)$ and so that $\chi$ is $C^1$ small (i.e., bounded by a universal constant times $\epsilon$) provided only that $P'$ is sufficiently close to $P$. Thus $h$ is holomorphic and close to $\varphi$ in the $C^1$ topology.

Now we set

$$\widetilde{h}(z) = \eta(z)\varphi(P + (z - P')) + (1 - \eta(z))\varphi(z) + \chi(z) - \chi(P')$$

and

$$\widetilde{\varphi}(z) = \frac{1}{1 + C \cdot \epsilon} \cdot \widetilde{h}\,.$$

We see that $\widetilde{\varphi}$ is holomorphic, it takes the value 0 at $P'$, and it is close to $\varphi$ in the $C^1$ topology. We may also conclude that $\widetilde{\varphi}$ shows that the Carathéodory metric at $P'$ in the direction $\xi' = \xi$ is $C^1$-close to the metric at $P$ in the direction $\xi$.

The same argument holds in reverse if we choose $P'$ as the base point at $P$ as the perturbed point. We may conclude therefore that the infinitesimal Carathéodory metric varies in the Lipschitz topology when the base point is perturbed.

A similar, but even easier, argument applies (just as in our discussion of the Kobayashi metric) when the tangent vector $\xi$ is perturbed. That completes our argument. □

The reader will note that we restricted the enunciation of this theorem to the case of strongly pseudoconvex domains. This is in marked contrast to the situation for the Kobayashi metric. The difference, of course, is that now we are considering maps *from* the domain $\Omega$ to the unit disc $D$. As a consequence, the $\overline{\partial}$ problem that we must solve lives on $\Omega$ rather than on the disc. We restrict to the strongly pseudoconvex case so that we may have favorable estimates on the $\overline{\partial}$ problem.

# 5 Concluding Remarks

It is a natural question to consider regularity of the infinitesimal invariant metrics. Our motivation was the study of extremal discs for the Kobayashi metric in the sense of Lempert [LEM]. But there are many contexts in which estimates of this kind may prove useful.

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