1.7.1a

\[ xy = c \]

Differentiate w.r.t. \( x \):

\[ 0 = \frac{d}{dx}(xy) = y + \frac{dy}{dx} x \quad \rightarrow \quad \frac{dy}{dx} = -\frac{y}{x} \]

**ODE corresponding to** \( xy = c \)

**ODE corresponding to orthogonal trajectories**:

\[ \frac{dy}{dx} = \frac{x}{y} \]

\[ \int y \, dy = \int x \, dx \]

\[ \frac{y^2}{2} = \frac{x^2}{2} + C \]

\[ \boxed{y^2 - x^2 = D} \]

This is the family of hyperbolas with asymptotes \( y = \pm x \).
1.71c
\[ x + y = c \]
\[ \frac{dy}{dx} \]

1 + \frac{dy}{dx} = 0
\frac{dy}{dx} = -1

so orthogonal trajectories
must solve \( \frac{dy}{dx} = 1 \)
\[ \int dy = \int dx \]
\[ y = x + c \]
find the ODE: \(2y \frac{dy}{dx} = 4c \rightarrow \frac{dy}{dx} = \frac{2c}{y}\)

we know that \(y^2 = 4c(x+c) \Rightarrow 4c^2 + (4x)c - y^2 = 0\)

solve for \(c\):
\[
c = \frac{-4x \pm \sqrt{16x^2 - 4(4)(-y^2)}}{2(4)}
\]
\[
= -x \pm \frac{\sqrt{x^2 + y^2}}{2}
\]

so ODE is
\[
\frac{dy}{dx} = -x \pm \frac{\sqrt{x^2 + y^2}}{y}
\]

continued...
diff. eq. of orthogonal trajectories is therefore

\[
\frac{dy}{dx} = \frac{-y}{-x \pm \sqrt{x^2 + y^2}} \cdot \frac{(x \mp \sqrt{x^2 + y^2})}{(-x \mp \sqrt{x^2 + y^2})}
\]

\[
= \frac{xy \mp y \sqrt{x^2 + y^2}}{x^2 - (x^2 + y^2)}
\]

\[
= \frac{xy \mp y \sqrt{x^2 + y^2}}{-y^2}
\]

\[
= \frac{-x \pm \sqrt{x^2 + y^2}}{y}, \text{ the same equation defining our original family.}
\]

We conclude that whenever 2 curves in this family intersect, they do so orthogonally.
1.7.6a
\[ y = x(\sin(x+c)) \quad \rightarrow \sin(x+c) = \frac{y}{x} \rightarrow c = \arcsin\left(\frac{y}{x}\right) \]
\[ \frac{dy}{dx} = \sin(x+c) + x\cos(x+c) \]
\[ = \frac{y}{x} + x\cos\left(\arcsin\left(\frac{y}{x}\right)\right) \]

1.7.6c
The family of circles is defined by
\[ (x-c)^2 + (y-c)^2 = c^2 \]
we can write \[ c^2 = x^2 - 2cx + c^2 + y^2 - 2yc + c^2 \]
\[ \rightarrow 0 = c^2 - (2x+2y)c + (x^2 + y^2) \]
so \[ c = \frac{2x+2y \pm \sqrt{(2x+2y)^2 - 4(x^2 + y^2)}}{2} = \frac{2x+2y \pm \sqrt{4xy}}{2} \]
\[ \rightarrow c = x + y \pm \sqrt{xy} \]
the ODE is \[ 2(x-c) + 2(y-c) \frac{dy}{dx} = 0 \quad \text{with} \quad c = x + y \pm \sqrt{xy} \]
1.8.1 b

\[ xy' - 3xy - 2y^2 = 0 \]

\[ x \frac{dy}{dx} = 3xy + 2y^2 \]

\[ \frac{1}{(3xy + 2y^2)} \frac{dx}{dx} - x \frac{dy}{dy} \]

**homogeneous of degree 2.**  **homogeneous of degree 1.**

Since the degrees of homogeneity are different, we can't apply the method.
\[ \frac{d}{dx} \left( x \sin \left( \frac{x}{x} \right) \right) = y \sin \left( \frac{x}{x} \right) + x \]

both homogeneous
of degree 1

\[ \frac{dy}{dx} \sin \left( \frac{y}{x} \right) = \frac{y}{x} \sin \left( \frac{y}{x} \right) + 1 \]

\[ z = \frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = x \frac{dz}{dx} + z \]

\[ \left( x \frac{dz}{dx} + z \right) \sin(z) = z \sin(z) + 1 \]

\[ x \frac{dz}{dx} + z = z + \frac{1}{\sin z} \]

\[ x \frac{dz}{dx} = \frac{1}{\sin z} \quad \Rightarrow \quad \sin(z) \, dz = \frac{1}{x} \, dx \]

\[ \Rightarrow -\cos(z) = \ln |x| + C \]

\[ \cos \left( \frac{y}{x} \right) = \ln \left| \frac{1}{x} \right| + D \]
We know \( ae \neq bd \) and we want to find \( h, k \) such that

\[
\frac{dw}{dz} = \frac{dy}{dx} = F \left( \frac{a(z-h) + b(w-k) + c}{d(z-h) + e(w-k) + f} \right) = F \left( \frac{az + bw}{dz + ew} \right).
\]

As we see this will work iff \( h, k \) satisfy

\[ ah + bk = c \quad \text{and} \quad dh + ek = f. \]

We have \( ah = c - bk \) and \( a(h + ek) = af \)

\[ d(ah) + aek = af, \]

so \( d(c - bk) + aek = af \)

\[ dc - bd k + aek = af \]

\[ (ae - bd) k = af - dc. \quad \text{Note} \quad ae - bd \neq 0, \text{so} \]

\[ k = \frac{af - cd}{ae - bd}. \]

Similarly, \( bk = c - ah \) and \( bf = b(dh + ek) = bdh + ebk \)

\[ = bdh + e(c - ah) = (bd - ae)h + ec, \quad \text{so} \]

\[ h = \frac{ce - bf}{ae - bd}. \]

(Note that we avoided dividing by anything we didn't know was nonzero.)
1.8.4a \[ \frac{dy}{dx} = \frac{x+y+4}{x-y-6} \]

Taking a hint from the previous problem, we would like to find \( h, k \) satisfying

\[ \begin{align*}
    h + k &= 4 \\
    h - k &= -6
\end{align*} \]

We can use the formula from that problem, or directly solve and see \( h = -1, \ k = 5 \).

So let \( x = z + 1, \ y = w - 5 \). Then

\[ \frac{dw}{dz} = \frac{z+1+w-5+4}{z+1-w+5-6} = \frac{z+w}{z-w} = \frac{1+\frac{w}{z}}{1-\frac{w}{z}}. \]

Now we make the substitution \( d = \frac{w}{z} \), \( \frac{dw}{dz} = z \frac{dd}{dz} + \alpha \). This gives

\[ 2 \frac{dd}{dz} + \alpha = \frac{1+\alpha}{1-\alpha} \]

\[ 2 \frac{dd}{dz} = \frac{1+\alpha}{1-\alpha} - \alpha (1-\alpha) = \frac{1+\alpha^2}{1-\alpha} \]

continued...
\[
\frac{1-x}{1+x^2} \, dx = \frac{1}{2} \, dz
\]

\[
\int \frac{1}{1+z^2} \, dz = \int \frac{1}{1+\frac{1}{2}z^2} \, dz = \int \frac{1}{2} \, dz
\]

\[\arctan(x) - \frac{1}{2} \ln(1+x^2) = \ln|z| + C\]

\[\arctan\left(\frac{w}{z}\right) - \frac{1}{2} \ln\left(1+\left(\frac{w}{z}\right)^2\right) = \ln|z| + C\]

\[\arctan\left(\frac{y+\frac{5}{x-1}}{x-1}\right) - \frac{1}{2} \ln\left(1+\left(\frac{y+\frac{5}{x-1}}{x-1}\right)^2\right) = \ln|x-1| + C\]

\[\arctan\left(\frac{y+\frac{5}{x-1}}{x-1}\right) - \ln\left(\frac{\frac{1}{x-1}}{\sqrt{1+\left(\frac{y+\frac{5}{x-1}}{x-1}\right)^2}}\right) = C\]
\[
\frac{1}{x} (xy - 1) \, dx + \frac{1}{x} (x^2 - xy) \, dy = 0
\]

So we have \( \frac{M_y - N_x}{N} = \frac{x - (2x - y)}{x^2 - xy} = -\frac{1}{x} \).

Let \( \mu(x) = e^{\int -\frac{1}{x} \, dx} = e^{\ln x} = \frac{1}{x} \).

\[
\frac{1}{x} (xy - 1) \, dx + \frac{1}{x} (x^2 - xy) \, dy = 0
\]

\[
(y - \frac{1}{x}) \, dx + (x - y) \, dy = 0 \text{ , exact.}
\]

\[
\int dy = \int dx
\]

1 = 1

so want \( F \) with \( F_x = y - \frac{1}{x} \), \( F_y = x - y \)

\( F_x = y - \frac{1}{x} \), so \( F = yx - \ln|x| + g(y) \), so \( F_y = x + g'(y) \).

So \( x + g'(y) = x - y \rightarrow g'(y) = -y \rightarrow g(y) = -\frac{y^2}{2} + C \).

Hence \( F = yx - \ln|x| - \frac{y^2}{2} + C \),

and \( yx - \ln|x| - \frac{y^2}{2} = D \) is an implicit solution.
\[
\frac{1.9.1e}{(x+2)\sin y \, dx + x\cos y \, dy = 0}
\]

Compute \( \frac{M_y - N_x}{N} = \frac{(x+2)\cos y - \cos y}{xcosy} = \frac{x+1}{x} = 1 + \frac{1}{x} \)

Try integrating factor \( \mu(x) = e^{\int 1/x \, dx} = xe^x \)

\[ xe^x (x+2)\sin y \, dx + x^2e^x \cos y \, dy = 0 \]

...you can check this is exact.
1.9.3

Let \( z = x + y \). Suppose there exists \( \mu(z) \) such that
\[
\mu(z) M(x, y) \, dx + \mu(z) N(x, y) = 0
\]
is exact. Then
\[
\frac{\partial}{\partial y} (\mu M) = \frac{\partial}{\partial x} (\mu N).
\]

We have
\[
\frac{\partial}{\partial y} \mu M = \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{d \mu}{d z} \frac{\partial \mu}{\partial y} \frac{\partial y}{\partial z} M + \mu \frac{\partial M}{\partial y} = \frac{d \mu}{d z} M + \mu \frac{\partial M}{\partial y}.
\]

Similarly,
\[
\frac{\partial}{\partial x} (\mu N) = \frac{d \mu}{d z} N + \mu \frac{\partial N}{\partial x}.
\]

Hence
\[
\frac{d \mu}{d z} M + \mu \frac{\partial M}{\partial y} = \frac{d \mu}{d z} N + \mu \frac{\partial N}{\partial x}, \quad \text{and}...
\]

continued...
\[ \frac{1}{\mu} \frac{1}{\frac{dx}{dz}} = \frac{\frac{2N}{3x} - \frac{2M}{3y}}{M - N}. \]

Therefore, \( \frac{2N}{3x} - \frac{2M}{3y} \) is a function of \( z \).

Conversely, suppose \( \frac{2N}{3x} - \frac{2M}{3y} \) is a function of \( z \). Call it \( g(z) \). Then the calculations above show that the solution \( \mu(z) \) to \( \frac{1}{\mu} \frac{1}{\frac{dx}{dz}} = g(z) \) will be an integrating factor.

So there is an integrating factor \( \mu(z) \) if \( \frac{2N}{3x} - \frac{2M}{3y} \) is a function of \( z \).