Solutions to Crowdmark HW 6
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12) \( y(0) = 0, y\left(\frac{\pi}{2}\right) = 0 \)
   \( y'' + \lambda y = 0 \)
   \[ y = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x \]
   \[ 0 = y(0) = A \text{, so } y = B \sin \sqrt{\lambda} x \]
   \[ 0 = y\left(\frac{\pi}{2}\right) = B \sin \left(\frac{\sqrt{\lambda}}{2}\right) \]
   \( \sqrt{\lambda} \) is an even integer, \( \sqrt{\lambda} = 2m \) for \( m = 1, 2, \ldots \)
   The eigenvalues are \( \lambda_m^2 = 4m^2 \)
   The eigenfunctions are \( \sin 2mx \).

\[ c) \quad y'' + \lambda y = 0 \]
\[ y(0) = 0, \quad y(1) = 0 \]
\[ y = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x \]
\[ 0 = y(0) = A \text{, so } y = B \sin \sqrt{\lambda} x \]
\[ 0 = y(1) = \sin \sqrt{\lambda} \text{ so } \sqrt{\lambda} = n\pi, \quad n = 1, 2, \ldots \]
   The eigenvalues are \( \lambda_n^2 = n^2 \pi^2 \).
   The eigenfunctions are \( \sin n\pi x \).

\[ f) \quad y'' + \lambda y = 0 \]
\[ y(2) = 0, \quad y(6) = 0 \]
\[ y = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x \]

\[ O = y (a) = A \cos (\sqrt{\lambda} a) + B \sin (\sqrt{\lambda} a) \]
\[ O = y (b) = A \cos (\sqrt{\lambda} b) + B \sin (\sqrt{\lambda} b) \]

\[ O = A \cos (\sqrt{\lambda} a) \cos (\sqrt{\lambda} b) + B \sin (\sqrt{\lambda} a) \cos (\sqrt{\lambda} b) \]
\[ O = A \cos (\sqrt{\lambda} a) \cos (\sqrt{\lambda} b) + B \sin (\sqrt{\lambda} b) \cos (\sqrt{\lambda} a) \]
\[ O = B \sin (\sqrt{\lambda} b) \cos (\sqrt{\lambda} a) - B \sin (\sqrt{\lambda} a) \cos (\sqrt{\lambda} b) \]
\[ \sin (\sqrt{\lambda} a) \cos (\sqrt{\lambda} b) = \sin (\sqrt{\lambda} b) \cos (\sqrt{\lambda} a) \]
\[ \tan (\sqrt{\lambda} a) = \tan (\sqrt{\lambda} b) \]

Thus \[ \sqrt{\lambda} b - \sqrt{\lambda} a = n \pi \] for \( n = 1, 2, \ldots \)

\[ \sqrt{\lambda} (b-a) = n \pi \]
\[ \sqrt{\lambda} = \frac{n \pi}{b-a} \]

\[ \lambda = \frac{n^2 \pi^2}{(b-a)^2} \]

So eigenvalues are \( \frac{n^2 \pi^2}{(b-a)^2} \)

Eigenvectors are \( A \cos \frac{n \pi}{b-a} x + B \sin \frac{n \pi}{b-a} x \).
2. \( \frac{\partial^2 y}{\partial x^2} = F''(x+zt) + G''(x-zt) \)

\( \frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial (x+zt)^2} + 4 \frac{\partial^2 y}{\partial (x-zt)^2} \).

So \( \frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial x^2} \). This is the wave equation.

b) \( \alpha = x + \alpha t \), \( \beta = x - \alpha t \)

\( \frac{\partial^2 y}{\partial \alpha \partial \beta} = ? \)

\( \frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} + \alpha \frac{\partial}{\partial \beta} = \frac{\partial}{\partial \alpha} + \frac{1}{\alpha} \frac{\partial}{\partial \beta} \)

\( \frac{\partial}{\partial \beta} = \frac{\partial}{\partial \alpha} - \frac{1}{\alpha} \frac{\partial}{\partial \beta} \)

\( \frac{\partial^2}{\partial \alpha^2} = \frac{1}{\alpha} \left( \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} \right) \)

\( \frac{\partial^2}{\partial \beta^2} = \frac{\alpha^2}{\alpha^2} \left( \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} \right) \)

\( \frac{\partial^2}{\partial t^2} = \alpha^2 \left( \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} \right) \)

So \( 0 = \left( 4 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \).
1. \[ w(x,t) = \sum b_j e^{-j^2 t} \sin jx + \left[w_1 + \frac{1}{\pi} (w_2 - w_1) x\right] \]

4. The new boundary conditions are:
   \[ \frac{\partial w}{\partial x}(0, t) = 0 \quad \forall t \]
   \[ \frac{\partial w}{\partial x}(\pi, t) = 0 \quad \forall t, \]

Now the temperature
   \[ w(x,t) = 100^\circ. \]

5. Now it is
   \[ w(x,t) = \text{average of } f \]
   \[ = \frac{1}{\pi} \int_0^\pi f(x) \, dx. \]

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1a) \[ f(\Theta) = \cos \frac{\Theta}{2}, -\pi \leq \Theta \leq \pi \]

\[ 2j^2 = \frac{4}{\pi^2} \int_0^\pi \cos \frac{\Theta}{2} \cos j\Theta \, d\Theta = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\Theta \cos 2j\Theta \, d\Theta \]

\[ = \frac{2}{\pi} \left( \cos(2j + 1)\pi - \cos(2j - 1)\pi \right) \frac{\pi}{2} \]

\[ = \frac{1}{\pi} \left[ \sin(2j + 1)\pi - \sin(2j - 1)\pi \right] \frac{\pi}{2} \]

\[ = \frac{1}{\pi} \left[ \frac{\sin(2j + 1)\pi}{2j + 1} - \frac{\sin(2j - 1)\pi}{2j - 1} \right] \frac{\pi}{2} \]
\[= \frac{1}{\pi} \left[ \frac{(\sin(j\pi - \frac{\pi}{2}) - \sin(j\pi + \frac{\pi}{2}))}{2j - 1} \right.\]
\[\left. - \frac{(\sin(-j\pi + \frac{\pi}{2}) - \sin(-j\pi - \frac{\pi}{2}))}{2j + 1} \right] \]
\[= \frac{1}{\pi} \left[ \frac{2\sin(j\pi - \frac{\pi}{2}) - 2\sin(j\pi + \frac{\pi}{2})}{2j - 1} \right.\]
\[\left. - \frac{2\sin(-j\pi + \frac{\pi}{2}) - 2\sin(-j\pi - \frac{\pi}{2})}{2j + 1} \right].\]

In case \( j \) is even, \( j = 2m \), then
\[x_j = \frac{2}{\pi} \left[ \frac{-1}{2j - 1} - \frac{1}{2j + 1} \right].\]
\[= \frac{2}{\pi} \left[ \frac{y_j^2 - 2}{4y_j^2 - 1} \right] = \frac{2}{\pi} \left[ \frac{y_j^2 - 2}{4y_j^2 - 1} \right].\]

In case \( j \) is odd, \( j = 2m - 1 \), then
\[x_j = \frac{2}{\pi} \left[ \frac{1}{2j - 1} - \frac{-1}{2j + 1} \right].\]
\[= \frac{2}{\pi} \left[ \frac{y_j^2 - 2}{4y_j^2 - 1} \right] = \frac{2}{\pi} \left[ \frac{y_j^2 - 2}{4y_j^2 - 1} \right].\]

\[b_j = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos \theta \sin j \theta \, d\theta = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos \theta \sin 2j \theta \, d\theta \]
\[= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \sin (2j+1) \theta + \sin (2j-1) \theta \, d\theta \]
\[= \frac{1}{\pi} \left[ \frac{-\cos (2j+1) \theta}{2j+1} - \frac{-\cos (2j-1) \theta}{2j-1} \right]_{-\pi/2}^{\pi/2} \]
\[= \frac{1}{\pi} \left[ \frac{-\cos (2j+1) \pi}{2j+1} - \frac{-\cos (2j-1) \pi}{2j-1} \right.\]
\[\left. - \frac{\cos (2j+1) (-\pi/2)}{2j+1} - \frac{\cos (2j-1) (-\pi/2)}{2j-1} \right] \]
\[= \frac{1}{\pi} \left[ \frac{-2\cos (2j+1) \pi}{2j+1} - \frac{-2\cos (2j-1) \pi}{2j-1} \right.\]
\[\left. - \frac{-2\cos (2j+1) (-\pi/2)}{2j+1} - \frac{-2\cos (2j-1) (-\pi/2)}{2j-1} \right].\]
In case \( j = 2m \) is even, \( b_j = \frac{2}{\pi} \left[ \frac{0}{2j+1} + \frac{0}{2j-1} \right] = 0 \).

In case \( j = 2m-1 \) is odd, \( b_j = \frac{2}{\pi} \left[ \frac{0}{2j+1} + \frac{0}{2j-1} \right] = 0 \).

So the Fourier series has only \( c_j \)'s in it.

The solution of the Dirichlet problem is (note \( a_0 = 0 \)).

\[
 w(r, \Theta) = \sum_{m=1}^{\infty} \frac{2m-1}{\pi (16m^2-1)} \cos 2m \Theta 
 + \sum_{m=2}^{\infty} \frac{8(2m-1)}{\pi (4(2m-1)^2-1)} \cos ((2m-1) \Theta).
\]

c) \( f(\Theta) = \begin{cases} 0, & -\pi \leq \Theta < 0 \\ \sin \Theta, & 0 \leq \Theta \leq \pi \end{cases} \)

\[
 j \geq 2 \quad a_j = \frac{1}{\pi} \int_{0}^{\pi} \sin \Theta \cos j \Theta \, d\Theta = \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin(1+j)\Theta + \sin(1-j)\Theta}{2} \, d\Theta 
 = \frac{1}{2\pi} \left[ \frac{-\cos(1+j)\Theta}{1+j} - \frac{-\cos(1-j)\Theta}{1-j} \right]_{0}^{\pi} 
 = \frac{1}{2\pi} \left[ \frac{-(-1)^{j+1}}{1+j} - \frac{(-1)^{j-1}}{1-j} \right] - \left( \frac{-1}{1+j} - \frac{1}{1-j} \right) 
 = \frac{1}{2\pi} \left[ \frac{(-1)^{j+1} - (-1)^{j-1}}{1-j^2} + \frac{2}{1-j^2} \right] 
 = \frac{1}{\pi}\left(1 - \frac{2}{1-j^2}\right) \cdot \left[ (-1)^{j+1} + 1 \right] 
 \]

If \( j = 2m \) is even, then \( a_j = \frac{2}{\pi (1-j^2)} \).

If \( j = (2m-1) \) is odd, then \( a_j = 0 \).

\[
 b_0 = \frac{1}{\pi} \int_{0}^{\pi} \sin \Theta \, d\Theta = \frac{1}{\pi} \left[ -\cos \Theta \right]_{0}^{\pi} = \frac{1}{\pi} \left[ 2 \right] = \frac{2}{\pi}.
\]
\[ j \geq 2 \quad b_j = \frac{1}{\pi j} \int_0^{\pi} \sin \theta \sin j \theta \cos \frac{\pi}{j} \theta - \cos \frac{\pi}{j+1} \theta \, d\theta \]
\[ = \frac{1}{2\pi j} \left[ \frac{\sin \left( \frac{\pi}{j} \theta \right)}{j-1} - \sin \left( \frac{\pi}{j+1} \theta \right) \right]_0^\pi \]
\[ = \frac{1}{2\pi j} \left[ \left( \frac{0}{j-1} - \frac{0}{j+1} \right) - \left( \frac{0}{j-1} - \frac{0}{j+1} \right) \right] = 0 \]
\[ j = 1 \quad b_1 = \frac{1}{\pi} \int_0^{\pi} \sin \theta \cos \theta \, d\theta = \frac{1}{\pi} \int_0^{\pi} \frac{1 - \cos 2\theta}{2} \, d\theta \]
\[ = \frac{1}{2} - \frac{\sin 2\theta}{4} \bigg|_0^\pi = (\frac{\pi}{2} - 0) - (0 - 0) = \frac{\pi}{2} \]

So, the solution of the Dirichlet problem is
\[ \sum_{m=1}^\infty \frac{2}{\pi (1 - 4m^2)^{\frac{3}{2}}} \cos 2m \theta + \frac{\pi}{2} \sin \theta. \]

d) \( \Phi(\theta) = \left\{ \begin{array}{ll} 0 & \text{if } -\pi \leq \theta < 0 \\ 1 & \text{if } 0 \leq \theta \leq \pi \end{array} \right. \)

\[ j \geq 1 \quad a_j = \frac{1}{\pi} \int_0^{\pi} \cos j \theta \, d\theta = \frac{1}{\pi} \left[ \sin j \theta \right]_0^\pi = 0 \]
\[ a_0 = \frac{1}{\pi} \int_0^{\pi} \, d\theta = 1 \]
\[ b_j = \frac{1}{\pi} \int_0^{\pi} \sin j \theta \, d\theta = \frac{1}{\pi} \left[ -\cos j \theta \right]_0^\pi \]
\[ = \frac{1}{\pi j} \left[ (-1)^{j+1} \right] \]

So, the solution of the Dirichlet problem is
\[ \frac{1}{2} + \sum_{j=1}^\infty \frac{(-1)^{j-1}}{\pi j} \left[ (-1)^{j+1} \right] \sin j \theta. \]
2. If \((r, \theta) = z\) be the variable in the unit disc, let \((r', \theta') = \hat{z}\) be the variable in \(D(0, R)\). Then \(\hat{z} = Rz\).

Let \(f(\theta)\) be the Dirichlet boundary data on \(\partial D(0, R)\).

Let \(w = \sum r^j (a_j \cos j\theta + b_j \sin j\theta)\) be the solution of the Dirichlet problem on \(D(0, 1)\) with boundary data \(f\).

Then \(\hat{w} = \sum \frac{r'^j}{r^j} (a_j \cos j\theta + b_j \sin j\theta)\) is a harmonic function on \(D(0, R)\) that has \(f\) as boundary function.

A similar change of variable gives the Poisson integral formula on \(D(0, R)\).

4. Done in class.
1. Now
\[ m(x) P(x) y'' + m(x) Q(x) y' + u(x) R(x) y = 0 \]
\[ = \left[ m(x) P(x) y' \right]' + \left[ S(x) y \right]' \]
\[ \therefore (m(x) P(x) - m(x) P(x)) y'' + (m(x) Q(x) - m(x) P(x)') y' \]
\[ + (m(x) R(x) - S(x)) y = 0 \]
\[ \therefore m(x) P(x) = m(x) P(x) \]
\[ m(x) Q(x) = (m(x) P(x))' + S(x) \]
\[ m(x) R(x) = S' (x) \]

Thus
\[ (m(x) P(x) y'' - (m(x) Q(x))' + m(x) R(x) y = 0 \]

So
\[ p(x) u''(x) + m'(x) (2 p(x)' - Q(x) + u(x) (P''(x) + R(x)) \]
\[ - Q'(x)) = 0 \]

(a) \( p = 1 - x^2 \), \( Q = -2x \), \( R = p (p + 1) \)
adjoint equation \( v \)
\[ (1 - x^2) u'' + (-2x) u' + (p (p + 1)) u = 0 \]
(b) \( p = x \), \( Q = x \), \( R = x^2 - p^2 \)
adjoint equation \( \tilde{v} \)
\[ x^2 \tilde{v}'' + 3x \tilde{v}' + [1 + x^2 - p^2] \tilde{v} = 0 \]
(c) \( P = 1, \ Q = -2x, \ R = 2y \)

Adjoint equation is

\[ u'' + 2xu' + (2+2p)u = 0 \]

(d) \( P = x, \ Q = 1-x, \ R = p \)

Adjoint equation is

\[ xu'' + (1+x)u' + (1+p)u = 0, \]

2. Take \( n = 1 \) and \( u(x) = 1 \). The Euler equation can be written as

\[ (x^2 y')' + (-xy)' = 0 \]

Integration gives

\[ x^2 y' - xy = C \]

\[ y' - \frac{1}{x} y = \frac{C}{x^2} \]

\[ \frac{1}{x} y' - \frac{1}{x^2} y = \frac{C}{x^3} \]

\[ (\frac{1}{x} y)' = \frac{C}{x^3} \]

\[ \int (\frac{1}{x} y)' \, dx = \int \frac{C}{x^3} \, dx \]

\[ \frac{1}{x} y = -\frac{C}{2x^2} + D \]

\[ y = -\frac{C}{2x} + Dx \]